CALCULUS I – REVIEW

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Abstract. The following document evolved from several "study guides" which were written to help Math 241 (Calculus I) students at the University of Hawaii at Manoa prepare for exams. The section references are for *University Calculus, Alternate Edition* by Hass, Weir, and Thomas.

Things to know

Precalculus (Prerequisite Material). Sections 1.1 – 1.3

- Functions, graphs, function composition $f \circ g$.

- Trigonometry, the six basic trigonometric functions. You should know the values of the trig functions on angles like $0, \pi/6, \pi/4, \pi/3, \pi/2$, and the corresponding angles in other quadrants.

Limits and Continuous Functions. Sections 2.2, 2.4, 2.5, 2.6

- To say $\lim_{x \to c} f(x) = L$ means that the values (the output) of the function $f(x)$ get closer and closer to the number L when we consider x (the input) closer and closer to c. The function f does not even have to be defined at $x = c$, and even if it is, the value $f(c)$ does not influence what the limit is. In other words, it doesn't matter what is happening *at* $x = c$; limits are all about what is happening *near* $x = c$.

- **The Limit Laws** Suppose that $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist. Then
  - ◇ $\lim_{x \to c} (f(x) + g(x)) = \left(\lim_{x \to c} f(x)\right) + \left(\lim_{x \to c} g(x)\right)$,
  - ◇ $\lim_{x \to c} (f(x) - g(x)) = \left(\lim_{x \to c} f(x)\right) - \left(\lim_{x \to c} g(x)\right)$,
  - ◇ $\lim_{x \to c} (f(x) \cdot g(x)) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right)$,
  - ◇ $\lim_{x \to c} (k \cdot f(x)) = k\left(\lim_{x \to c} f(x)\right)$ for any constant k,
  - ◇ $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, provided $\lim_{x \to c} g(x) \neq 0$,
  - ◇ $\lim_{x \to c} (f(x))^r = \left(\lim_{x \to c} f(x)\right)^r$, for any exponent r, as long as the expression makes sense. (Notice it wouldn't make sense if $\lim_{x \to c} f(x) < 0$ and $r = 1/2$.)

- Limits of polynomial functions are evaluated by *substitution*. That is, if $p(x)$ is a polynomial function, then

$$\lim_{x \to c} p(x) = p(c).$$

This is because polynomial functions are continuous.

- For a limit of a rational function $\frac{f(x)}{g(x)}$, if $g(c) \neq 0$ then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}.$$

(This is the easy case.) However, if $g(c) = 0$, something else needs to be done. Usually one can try to factor and cancel factors, e.g.

$$\lim_{x \to 3} \frac{x^2 - 9}{x^2 - 5x + 6} = \lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)(x-2)} = \lim_{x \to 3} \frac{x+3}{x-2} = \frac{3+3}{3-2} = 6.$$

- One can usually *rationalize* to compute tricky limits with square roots, e.g.

$$\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{4+x} - 2}{x} &= \lim_{x \to 0} \left(\frac{\sqrt{4+x} - 2}{x} \cdot \frac{\sqrt{4+x} + 2}{\sqrt{4+x} + 2} \right) \\
&= \lim_{x \to 0} \frac{(4+x) - 4}{x(\sqrt{4+x} + 2)} \\
&= \lim_{x \to 0} \frac{x}{x(\sqrt{4+x} + 2)} \\
&= \lim_{x \to 0} \frac{1}{\sqrt{4+x} + 2} \\
&= \frac{1}{\sqrt{4+0} + 2} = \frac{1}{4}.
\end{aligned}$$

- The **Sandwich Theorem** states that if there are three functions f, g, h such that

$$g(x) \leq f(x) \leq h(x) \qquad \text{for all } x \text{ near } c,$$

and if

$$\lim_{x \to c} g(x) = L \quad \text{and} \quad \lim_{x \to c} h(x) = L,$$

then we can conclude that $\lim_{x \to c} f(x) = L$ also. A picture of the graphs of three such functions makes it easier to understand why this is true (see pictures in the text).

- Know, and have respect for (cause it's not easy), the limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

We also have the generalization that

$$\lim_{x \to 0} \frac{\sin(ax)}{ax} = 1 \qquad \text{for any constant } a.$$

We can *use* this result to compute other limits through algebraic manipulation, e.g.

$$\begin{aligned}\lim_{x\to 0}\frac{\tan(2x)}{x} &= \lim_{x\to 0}\frac{\sin(2x)}{x\cos(2x)}\\ &= \lim_{x\to 0}\left(\frac{\sin(2x)}{x}\cdot\frac{1}{\cos(2x)}\right)\\ &= \lim_{x\to 0}\left(2\cdot\frac{\sin(2x)}{2x}\cdot\frac{1}{\cos(2x)}\right)\\ &= 2\left(\lim_{x\to 0}\frac{\sin(2x)}{2x}\right)\left(\lim_{x\to 0}\frac{1}{\cos(2x)}\right)\\ &= 2\cdot 1\cdot\frac{1}{\cos(0)} = 2.\end{aligned}$$

- For the one-sided limit $\lim_{x\to c^-} f(x)$, we only consider x which are *less than* c, whereas for the one-sided limit $\lim_{x\to c^+} f(x)$, we only consider x which are *greater than* c.
- The two-sided limit $\lim_{x\to c} f(x)$ exists if and only if both one-sided limits $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist and are the same value.
- We say $\lim_{x\to\infty} f(x) = L$ if whenever we consider larger and larger and larger values of x (e.g. $x = 10^3, 10^6, 10^{10}, 10^{1478}, 10^{234987235}$, and so on...) the values of the function $f(x)$ get closer and closer to the number L. Similarly the limit $\lim_{x\to-\infty} f(x)$ is about what happens for larger and larger *negative* values of x (e.g. $x = -10^3, -10^6, -10^{10}, -10^{1478}, -10^{234987235}$, and so on...) Graphically, if $\lim_{x\to\infty} f(x) = L$, then the horizontal line $y = L$ will be a horizontal asymptote for the graph of the function $y = f(x)$. Similarly, if $\lim_{x\to-\infty} f(x) = M$, then the line $y = M$ will be a horizontal asymptote.
- For any positive number p,

$$\lim_{x\to\infty}\frac{1}{x^p} = 0 \quad\text{and}\quad \lim_{x\to-\infty}\frac{1}{x^p} = 0.$$

This can be used, along with algebraic manipulation, to calculate limits of rational functions as $x\to\pm\infty$. When faced with such a limit, divide the numerator and denominator by the largest power of x in the denominator, e.g.

$$\begin{aligned}\lim_{x\to\infty}\frac{4x+2x^2-7}{\frac{x^2}{5}+1000x} &= \lim_{x\to\infty}\frac{(4x+2x^2-7)}{(\frac{x^2}{5}+1000x)}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\\ &= \lim_{x\to\infty}\frac{\frac{4}{x}+2-\frac{7}{x^2}}{\frac{1}{5}+\frac{1000}{x}}\\ &= \frac{0+2-0}{\frac{1}{5}+0} = 10.\end{aligned}$$

- We say that $\lim_{x\to c} f(x) = \infty$ if as the input x gets closer and closer to c, the values $f(x)$ grow larger and larger without bound. Similarly, we say $\lim_{x\to c} f(x) = -\infty$ if the values $f(x)$ are larger and larger *negative* numbers (without bound). If $\lim_{x\to c} = \infty$ or $-\infty$, then the vertical line $x = c$ will be a vertical asymptote for the graph $y = f(x)$.

- Infinite limits will occur for a limit of the form $\lim_{x\to c} \frac{f(x)}{g(x)}$ with the property that
  (1) $\lim_{x\to c} g(x) = 0$, and
  (2) $\lim_{x\to c} f(x)$ is **some finite nonzero number**.

  In this case, the limit will be either ∞ or $-\infty$. One must pay careful attention to the signs ($+$ or $-$) in the expression to determine whether the answer is ∞ or $-\infty$. For example, consider the limit
  $$\lim_{x\to 3^+} \frac{x-1}{3x^2 - x^3}.$$
  Notice the limit of the denominator is 0 and the limit of the numerator is 2 (a finite, nonzero number!). This indicates the limit will be either ∞ or $-\infty$. Let's factor the expression and consider the signs:
  $$\lim_{x\to 3^+} \frac{x-1}{3x^2 - x^3} = \lim_{x\to 3^+} \frac{x-1}{x^2(3-x)}$$
  We are supposed to consider x that are slightly larger than 3. For such an x, the numerator $x - 1$ will be positive (it's close to 2, so its definitely positive). The x^2 in the denominator is certainly positive, because a square is never negative. The last factor $3 - x$ is where we must be careful. If x is a little bigger than 3, then $3 - x$ will be close to 0, but it will ever so slightly be *less than* 0, making it *negative*. So
  $$\text{the signs of } \frac{x-1}{x^2(3-x)} \text{ look like } \frac{(-)}{(+)(+)}$$
  when x is a little bigger than 3. The overall sign is negative, and we conclude
  $$\lim_{x\to 3^+} \frac{x-1}{x^2(3-x)} = -\infty.$$
  In a similar way, we can determine
  $$\lim_{x\to 3^-} \frac{x-1}{3x^2 - x^3} = \infty.$$

- In contrast to the previous point, if a limit $\lim_{x\to c} \frac{f(x)}{g(x)}$ is such that
  (1) $\lim_{x\to c} g(x) = 0$, and
  (2) $\lim_{x\to c} f(x) = 0$,

  then you can probably cancel factors or rationalize, see the previous discussions.

- A function f is called *continuous at the point $x = c$* if

$$\lim_{x\to c} f(x) = f(c).$$

If f is continuous at every point, then f is called a *continuous function*. Intuitively, a function is continuous if you can draw its graph without ever lifting your pencil, though this is not a rigorous definition. Many functions are continuous:
  - ⋄ Polynomial functions are continuous.
  - ⋄ A rational function $\frac{f(x)}{g(x)}$ is continuous at any c for which $g(c) \neq 0$.
  - ⋄ Sums, differences, products, powers, and compositions of continuous functions are continuous.
  - ⋄ A quotient $\frac{f(x)}{g(x)}$ of continuous functions is continuous at any c for which $g(c) \neq 0$.
  - ⋄ The trig functions $\sin x$ and $\cos x$ are continuous.
  - ⋄ The other trig functions

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

are continuous at all points where their denominators are not zero.

- The **Intermediate Value Theorem** states that if f is a continuous function on a closed interval $[a, b]$, then every value between the numbers $f(a)$ and $f(b)$ is in the range of f. That is, for every number c between $f(a)$ and $f(b)$, there is some x in the interval $[a, b]$ such that $f(x) = c$.

As an example, let's use this theorem to prove that there is a solution x to the equation

$$x^5 - 4x + 1 = 0$$

which lies in the closed interval $[0, 1]$. Notice that $f(x) = x^5 - 4x + 1$ is a continuous function. Moreover, we can calculate that

$$f(0) = 0^5 - 4(0) + 1 = 1, \qquad \text{and} \qquad f(1) = 1^5 - 4(1) + 1 = -2.$$

The Intermediate Value Theorem says that every value between 1 and -2 will be part of the range of f over the interval $[0, 1]$. So there must be some x in $[0, 1]$ with the property that $f(x) = 0$. That is, this x satisfies

$$x^5 - 4x + 1 = 0.$$

Derivatives. Sections 2.1, 2.7, 3.1

- Let $y(t)$ be the distance an object traveled as a function of time. The *average speed* of the object between $t = a$ and $t = b$ is given by

$$\text{average speed} = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{y(b) - y(a)}{b - a}.$$

Let's focus on a specific time $t = a$, and estimate the speed at this time by choosing a length of a time interval, which we call h, and calculating the average speed between $t = a$ and $t = a + h$. Here,

$$\text{average speed} = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{y(a+h) - y(a)}{(a+h) - a} = \frac{y(a+h) - y(a)}{h}.$$

Intuitively, this approximation to our actual speed should get better and better the smaller h gets. Thus if we take a limit as $h \to 0$, we obtain

$$\text{exact speed at time } a = \lim_{h \to 0} \frac{y(a+h) - y(a)}{h}.$$

- Given any function $f(x)$, we can talk about the *average rate of change* of f over an interval $[a, b]$ in an analogous way:

$$\text{average rate of change of } f = \frac{f(b) - f(a)}{b - a}.$$

This value corresponds to the slope of the secant line, i.e. the line connecting the two points on the graph of $y = f(x)$ corresponding to $x = a$ and $x = b$. We could also take the point of view of calculating the average rate of change starting at some a for an interval of length h. The average rate of change of f on $[a, a + h]$ is

$$\text{average rate of change of } f = \frac{f(a+h) - f(a)}{h}.$$

If we shrink h to 0, we obtain the *instantaneous rate of change of f at a*

$$\text{instantaneous rate of change of } f \text{ at } a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

This is more commonly called the *derivative of f at a*, and is denoted $f'(a)$. This has the interpretation as the slope of the tangent line to the graph $y = f(x)$ at the point $x = a$. To see this, visualize the secant line for the interval $[a, a + h]$ and imagine how it will slide along the picture when we shrink h to 0. In the limit, it is exactly the tangent line.

- The definition of the derivative of a function f is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

In general, this limit may not exist. The function f is called *differentiable at a point* x if the limit $f'(x)$ exists. A function is just called *differentiable* if it is differentiable at every point x.

The derivative $f'(x)$ of a function is another function. The way the function $f'(x)$ works is that it takes an input x, and the output is the slope of the tangent line to the graph of f at the point x. (Notice different choices of x will have different tangent lines, and therefore different slopes.)

- As an example, let's calculate the derivative of $f(x) = x^2$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} (2x + h) \\ &= 2x + 0 = 2x. \end{aligned}$$

  This shows $\frac{d}{dx}\left[x^2\right] = 2x$.

- Every differentiable function is also continuous. Graphically, the distinction between the two is that the graph of a differentiable function will never have sharp corners or cusps (it must always have a nice tangent line everywhere). So not every continuous function is differentiable. Everyone's favorite example is the absolute value function $f(x) = |x|$. It's graph makes a "V" shape, which has a sharp corner at $x = 0$. The absolute value function is continuous, but it is not differentiable at $x = 0$ (notice there's no reasonable tangent line at $x = 0$).

- Recall that the equation of the line through the point (a, b) with slope m is given by

$$y - b = m(x - a).$$

  Using this along with derivatives, we can give the equation of the tangent line to the graph $y = f(x)$ at a given x value a. The corresponding point on the graph, and hence on the tangent line, is $(a, f(a))$, and the slope of the tangent line is given by the number $f'(a)$. Putting it all together, the tangent line at a will have equation

$$y - f(a) = f'(a)(x - a).$$

Differentiation Rules. Sections 3.2, 3.4, 3.5

- The derivative of any constant function is 0. For example,

$$\frac{d}{dx}\Big[3\Big] = 0 \quad \text{and} \quad \frac{d}{dx}\Big[-7\Big] = 0.$$

  Notice the graph of any constant function is a horizontal line. Thus the tangent line at any point is also a horizontal line, and so has slope 0.

- **Power Rule:** For any real number n,

$$\frac{d}{dx}\Big[x^n\Big] = nx^{n-1}.$$

  For example,

$$\frac{d}{dx}\Big[x^3\Big] = 3x^2 \quad \text{and} \quad \frac{d}{dx}\Big[x^{14}\Big] = 14x^{13}.$$

Powers of x in the denominator are negative powers of x, as in

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}\left[x^{-1}\right] = (-1)x^{-2} = -\frac{1}{x^2},$$

and

$$\frac{d}{dx}\left[\frac{1}{x^5}\right] = \frac{d}{dx}\left[x^{-5}\right] = (-5)x^{-6} = -\frac{5}{x^6}.$$

The exponent can be a fraction, as in

$$\frac{d}{dx}\left[x^{8/3}\right] = \frac{8}{3}x^{5/3},$$

and

$$\frac{d}{dx}\left[\sqrt{x}\right] = \frac{d}{dx}\left[x^{1/2}\right] = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

The exponent can be any real number (as long as its a constant), like

$$\frac{d}{dx}\left[x^{\pi}\right] = \pi x^{\pi-1} \quad \text{and} \quad \frac{d}{dx}\left[x^{\sqrt{2}}\right] = \sqrt{2}x^{\sqrt{2}-1}.$$

- **Sum Rule:** For any differentiable functions f and g,

$$\frac{d}{dx}\Big[f(x)+g(x)\Big] = \frac{d}{dx}\Big[f(x)\Big] + \frac{d}{dx}\Big[g(x)\Big].$$

This says that when differentiating a sum, we can just differentiate *term-by-term*. For example,

$$\frac{d}{dx}\left[x^4+\sin x\right] = \frac{d}{dx}\left[x^4\right] + \frac{d}{dx}\left[\sin x\right] = 4x^3 + \cos x.$$

- **Constant Multiple Rule:** For any constant c and differentiable function f,

$$\frac{d}{dx}\Big[cf(x)\Big] = c\frac{d}{dx}\Big[f(x)\Big].$$

The slogan is that "constants come along for the ride." They don't really affect the differentiation process, but you can't forget them. For example,

$$\frac{d}{dx}\left[6x^{3/2}\right] = 6\frac{d}{dx}\left[x^{3/2}\right] = 6\cdot\frac{3}{2}x^{1/2} = 9x^{1/2}$$

and

$$\frac{d}{dx}\Big[-\tan x\Big] = \frac{d}{dx}\Big[(-1)\tan x\Big] = (-1)\frac{d}{dx}\Big[\tan x\Big] = -\sec^2 x.$$

- **Product Rule:** For any differentiable functions f and g,

$$\frac{d}{dx}\Big[f(x)g(x)\Big] = \frac{d}{dx}\Big[f(x)\Big]g(x) + f(x)\frac{d}{dx}\Big[g(x)\Big].$$

Some examples are

$$\frac{d}{dx}\left[x^2\sin x\right] = \frac{d}{dx}\left[x^2\right]\sin x + x^2\frac{d}{dx}\left[\sin x\right] = 2x\sin x + x^2\cos x$$

and

$$\frac{d}{dx}\Big[\sec x \tan x\Big] = \frac{d}{dx}\Big[\sec x\Big]\tan x + \sec x \frac{d}{dx}\Big[\tan x\Big]$$
$$= (\sec x \tan x)\tan x + \sec x(\sec^2 x)$$
$$= \sec x \tan^2 x + \sec^3 x.$$

If you have a product with three factors, the product rule looks like

$$\frac{d}{dx}\Big[f(x)g(x)h(x)\Big] = \frac{d}{dx}\Big[f(x)\Big]g(x)h(x) + f(x)\frac{d}{dx}\Big[g(x)\Big]h(x) + f(x)g(x)\frac{d}{dx}\Big[h(x)\Big].$$

One can prove this by iterating the regular product rule twice. Similar formulas hold for products or four or five or six (and so on) factors.

- **Quotient Rule:** For any differentiable functions f and g,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}\big[f(x)\big]g(x) - f(x)\frac{d}{dx}\big[g(x)\big]}{[g(x)]^2}.$$

For example,

$$\frac{d}{dx}\left[\frac{2x^2+5}{x^3+4x+1}\right] = \frac{\frac{d}{dx}\left[2x^2+5\right](x^3+4x+1) - (2x^2+5)\frac{d}{dx}\left[x^3+4x+1\right]}{(x^3+4x+1)^2}$$
$$= \frac{(4x)(x^3+4x+1) - (2x^2+5)(3x^2+4)}{(x^3+4x+1)^2}$$
$$= \frac{4x^4+16x^2+4x - (6x^4+23x^2+20)}{(x^3+4x+1)^2}$$
$$= \frac{-2x^4-7x^2+4x-20}{(x^3+4x+1)^2}.$$

Pretty crazy that this quantity tells you the slopes of the tangent lines to the graph of $y = \dfrac{2x^2+5}{x^3+4x+1}$, isn't it? As another example,

$$\frac{d}{dx}\Big[\tan x\Big] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]$$
$$= \frac{\frac{d}{dx}\big[\sin x\big]\cos x - \sin x\frac{d}{dx}\big[\cos x\big]}{(\cos x)^2}$$
$$= \frac{(\cos x)\cos x - \sin x(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x}$$
$$= \sec^2 x.$$

- **Chain Rule:** For any differentiable functions f and g,

$$\frac{d}{dx}\Big[f(g(x))\Big] = f'(g(x))g'(x).$$

One common occurrence of this is when f is a trigonometric function, like

$$\frac{d}{dx}\Big[\cos(x^2+x)\Big] = -\sin(x^2+x)\frac{d}{dx}\Big[x^2+x\Big] = -\sin(x^2+x)\cdot(2x+1),$$

or

$$\frac{d}{dx}\Big[\sec(\sin x)\Big] = \sec(\sin x)\tan(\sin x)\frac{d}{dx}\Big[\sin x\Big] = \sec(\sin x)\tan(\sin x)\cos x.$$

Another common occurrence of the chain rule is when the outside function f is of the form $f(x) = x^n$. Then we obtain the **Power Chain Rule:**

$$\frac{d}{dx}\Big[g(x)^n\Big] = ng(x)^{n-1}g'(x).$$

For example,

$$\frac{d}{dx}\Big[\sqrt{\sin x}\Big] = \frac{d}{dx}\Big[(\sin x)^{1/2}\Big] = \frac{1}{2}(\sin x)^{-1/2}\frac{d}{dx}\Big[\sin x\Big] = \frac{\cos x}{2\sqrt{\sin x}}$$

or

$$\frac{d}{dx}\Big[(1+x^2)^{-100}\Big] = (-100)(1+x^2)^{-101}\frac{d}{dx}\Big[1+x^2\Big] = -200x(1+x^2)^{-101}.$$

Sometimes multiple applications of the chain rule are necessary:

$$\begin{aligned}\frac{d}{dx}\Big[\sin\big(x+\cos(x^2)\big)\Big] &= \cos\big(x+\cos(x^2)\big)\frac{d}{dx}\Big[x+\cos(x^2)\Big] \\ &= \cos\big(x+\cos(x^2)\big)\left(1+\frac{d}{dx}\Big[\cos(x^2)\Big]\right) \\ &= \cos\big(x+\cos(x^2)\big)\left(1+(-\sin(x^2))\frac{d}{dx}\Big[x^2\Big]\right) \\ &= \cos\big(x+\cos(x^2)\big)\big(1-2x\sin(x^2)\big).\end{aligned}$$

- The derivatives of the six trigonometric functions are

$$\begin{aligned} \frac{d}{dx}\Big[\sin x\Big] &= \cos x & \frac{d}{dx}\Big[\cos x\Big] &= -\sin x \\ \frac{d}{dx}\Big[\tan x\Big] &= \sec^2 x & \frac{d}{dx}\Big[\cot x\Big] &= -\csc^2 x \\ \frac{d}{dx}\Big[\sec x\Big] &= \sec x\tan x & \frac{d}{dx}\Big[\csc x\Big] &= -\csc x\cot x \end{aligned}$$

Observe the symmetry between the two columns. It can save some time to remember all six of these. However, notice that if you know the derivative of $\sin x$ and $\cos x$, you recover the other four using the quotient rule (as shown above for $\tan x$).

More with Derivatives. Sections 3.2, 3.3

- The **second derivative** of a function f is the derivative of the derivative, that is

$$f''(x) = (f')'(x) = \frac{d}{dx}\Big[f'(x)\Big].$$

The second derivative of $y = f(x)$ is also denoted $\dfrac{d^2y}{dx^2}$. The derivative of the second derivative is the **third derivative**, and so on. In general, the n**-th derivative** of $y = f(x)$ is denoted

$$f^{(n)}(x) \quad \text{or} \quad \frac{d^ny}{dx^n}.$$

- Derivatives are **rates of change**. The quantity dy/dx approximates how much the quantity y will change per unit change in x (see also linear approximation and differentials below). This is easiest to understand intuitively when the independent variable is time t. Suppose y is some physical quantity that is changing with time t. For example, say $y(t)$ is the temperature in degrees (say, Fahrenheit) of a cup of coffee after t minutes. Then dy/dt is the rate at which the temperature of the coffee is changing. To determine the units for the derivative, think of dy/dt as a quotient. The units in the numerator are degrees and the units in the denominator are minutes. Thus, the units for dy/dt are degrees/minute. So if $dy/dt = -5$, then at this moment in time, the coffee is cooling at a rate of 5 degrees per minute (it's cooling because the derivative is negative, which means y is decreasing). This does not necessarily mean that the coffee will be exactly 5 degrees cooler one minute later, because the instantaneous rate of change dy/dt could also be changing during this minute. However, it makes sense to say the coffee should be *about* 5 degrees cooler (this is a linear approximation).

- **Motion along a line:** Suppose an object is moving along an axis. Let $s(t)$ denote its coordinate (or **position**) on the axis at time t. The rate at which position changes with respect to time is **velocity**. That is, velocity is the function
$$v(t) = \frac{ds}{dt}.$$
The sign of the velocity indicates in which of the two possible directions the object is traveling. **Speed** is the absolute value (or magnitude) of the velocity
$$\text{Speed } = \left|\frac{ds}{dt}\right|.$$
So speed is velocity in which we forgot the direction. The rate at which velocity changes with respect to time is **acceleration**
$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Implicit Differentiation. Section 3.6

- When a variable y is explicitly given as a function of another variable x, that is,
$$y = f(x),$$
then we have many rules about indicating how to find the derivative $\dfrac{dy}{dx}$. Sometimes the relationship between two variables x and y is instead given

implicitly by an equation such as

$$x^2 + y^2 = 1.$$

The set of points satisfying this particular equation is the unit circle, and it makes sense to ask about $\frac{dy}{dx}$, the slope of tangent lines to the circle. There are two ways to attack this problem. The first approach (which is somewhat inadequate in other situations) is to solve for y explicitly as a function of x. Here we obtain

$$y = \pm\sqrt{1-x^2}.$$

Notice this is actually two functions

$$y = \sqrt{1-x^2} \quad \text{and} \quad y = -\sqrt{1-x^2}.$$

The graph of the first is the top half of the circle and the graph of the second is the bottom half of the circle. We can differentiate these two functions to obtain (you should double-check these calculations)

$$\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}} \quad \text{and} \quad \frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}}.$$

The first formula applies to points on the top half of the circle and the second applies to points on the bottom half of the circle. Using this, we can calculate slopes of tangent lines to the circle.

A second approach (which is actually simpler) is to return to the original equation

$$x^2 + y^2 = 1$$

and differentiate **implicitly**. The way we do this is we view y as a function of x, and then differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}\left[1\right].$$

To differentiate y^2 *with respect to* x, we need the chain rule, specifically the power chain rule (remember we are viewing y as a function of x). So

$$\frac{d}{dx}\left[y^2\right] = 2y\frac{dy}{dx}.$$

So the equation above becomes

$$2x + 2y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = -\frac{x}{y}.$$

In implicit differentiation, the derivative is usually expressed in terms of both x and y. As an application, let's find the slope of the tangent line to the unit circle at the point $(\frac{\sqrt{3}}{2}, \frac{1}{2})$. We can just plug in the two coordinates into our derivative formula to obtain

$$\frac{dy}{dx} = -\frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} = -\sqrt{3}.$$

For more complicated examples like

$$x^2 \cos^2 y = \sin y,$$

the first approach of solving for y in terms of x seems impossible. Let's just differentiate implicitly to find $\frac{dy}{dx}$:

$$\frac{d}{dx}\Big[x^2 \cos^2 y\Big] = \frac{d}{dx}\Big[\sin y\Big].$$

Recall that y is viewed as a function of x, so that chain rule is needed when dealing with y terms. On the left:

$$\begin{aligned}\frac{d}{dx}\Big[x^2 \cos^2 y\Big] &= \frac{d}{dx}\Big[x^2\Big]\cos^2 y + x^2 \frac{d}{dx}\Big[\cos^2 y\Big] \\ &= (2x)\cos^2 y + x^2(2\cos y)(-\sin y)\frac{dy}{dx},\end{aligned}$$

and on the right:

$$\frac{d}{dx}\Big[\sin y\Big] = \cos y \frac{dy}{dx}.$$

Inserting these back into our equation,

$$2x\cos^2 y - 2x^2 \cos y \sin y \frac{dy}{dx} = \cos y \frac{dy}{dx}..$$

We can cancel a common factor of $\cos y$:

$$2x\cos y - 2x^2 \sin y \frac{dy}{dx} = \frac{dy}{dx}.$$

Now rearrange to group together the $\frac{dy}{dx}$ terms and move everything else to the other side:

$$\frac{dy}{dx} + 2x^2 \sin y \frac{dy}{dx} = 2x\cos y.$$

Factor out the $\frac{dy}{dx}$:

$$\left(1 + 2x^2 \sin y\right)\frac{dy}{dx} = 2x\cos y,$$

and solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{2x\cos y}{1 + 2x^2 \sin y}.$$

Linear Approximation and Differentials. Section 3.8

- The **linear approximation** (or **linearization**) of a function $f(x)$ centered at a value $x = a$ is the function $L(x)$ whose graph is the tangent line to $y = f(x)$ at $x = 1$. From these, one can derive the explicit formula

$$L(x) = f(a) + f'(a)(x - a).$$

Since a is a concrete number (its something given), this is a linear function, meaning there are only x terms and a constant term. For x values near a, the tangent line looks a lot like the graph of the curve $y = f(x)$. Thus we obtain the linear approximation

$$f(x) \approx L(x) \quad \text{for } x \text{ near } a.$$

That is,

$$f(x) \approx f(a) + f'(a)(x - a) \quad \text{for } x \text{ near } a.$$

The linear function $L(x)$ is typically much easier to work with than the function $f(x)$ from which it came. One can use $L(x)$ to approximate values of the function $f(x)$, as we did in class with square roots.

As an example, lets calculate the linearization of $f(x) = \sqrt[4]{x}$ at $a = 16$ and use it to approximate $\sqrt[4]{18}$ by hand. The derivative is

$$f'(x) = \frac{d}{dx}\left[x^{1/4}\right] = \frac{1}{4}x^{-3/4} = \frac{1}{4(\sqrt[4]{x})^3}.$$

So at $a = 16$, we have

$$f(16) = \sqrt[4]{16} = 2 \quad \text{and} \quad f'(16) = \frac{1}{4(\sqrt[4]{16})^3} = \frac{1}{4(2)^3} = \frac{1}{32}.$$

The linearization is

$$L(x) = f(16) + f'(16)(x - 16) = 2 + \frac{1}{32}(x - 16).$$

We have

$$\sqrt[4]{18} \approx L(18) = 2 + \frac{1}{32}(18 - 16) = 2 + \frac{1}{16} = \frac{33}{16}.$$

How good was our approximation? Well, our approximation was

$$\sqrt[4]{18} \approx \frac{33}{16} = 2.0625,$$

whereas the true value (found by calculator) is

$$\sqrt[4]{18} = 2.059767\ldots$$

Our approximation was pretty good, as the error was less than 0.003. Remember that the approximation is better if x is chosen closer to a. So we would have a smaller error than the above if we approximated $\sqrt[4]{17}$ instead.

- Given a function $y = f(x)$, the **differential formula** is

$$dy = f'(x)dx.$$

For example, if $y = x^3$, then

$$dy = 3x^2dx.$$

The elements dx and dy are called **differentials**. Numerically, they are supposed to represent changes in the x-value and y-value respectively. The differential dx is viewed as an independent variable, and the differential dy is viewed as a dependent variable, which depends on dx and also x. If we change the x value by the amount dx, then the differential formula tells us how much y will change, approximately. The differential formula is based on linear approximation, so it does not give exact results.

As an example, suppose the radius of a circle increases from 10 m to 10.1 m. By about how much does the area increase? Notice that area of a circle is a function of radius:

$$A = \pi r^2.$$

The corresponding differential formula is

$$dA = (2\pi r)dr.$$

This change is represented by the fact that $r = 10$ and $dr = 0.1$. So the corresponding change in area is approximated by

$$dA = (2\pi r)dr = (2\pi(10))(0.1) = 2\pi.$$

The circle gains about $2\pi\,\mathrm{m}^2$ units of area.

Related Rates. Section 3.7

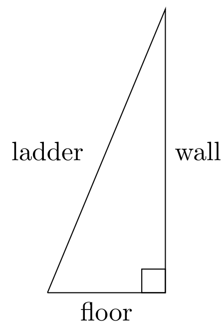
- In a **Related Rates problem**, one is presented with a situation that is changing with time. Partial information is given about certain quantities as well as their rates of change. One needs to use this information to deduce the rate of change of another quantity. Here is a general strategy:
  (1) **Draw a picture of the situation.** Identify the quantities in the picture that are changing with time, and assign them variable names. These variables are all functions of time t. If a certain quantity is constant throughout the entire process, you don't need to give it a name.
  (2) **Collect the given information from the problem and express it all in terms of the symbols you chose.** Recall that the rate at which a quantity changes is the value of its derivative. Remember also that if the quantity is increasing, its derivative is positive, and if it's decreasing, its derivative is negative.
  (3) **Determine which quantity you are being asked to find.** It's typically a rate of change, that is, the value of a derivative.
  (4) **Find a relationship (i.e. equation) between the variables in your problem.** This is a crucial step because if you want to relate the rates to each other, you first need to relate the variables to each other. This often comes from geometry (so draw a picture!): for example the Pythagorean Theorem, trigonometric formulas, formulas for areas/volumes of geometric objects, similar triangles, etc.
  (5) **(Implicitly) Differentiate your equation with respect to time t.** This step provides the equation which allows you to relate the rates to each other. Recall that all your variables are functions of time t, so you will likely need the chain rule, e.g.

  $$\frac{d}{dt}\left[s^2\right] = 2s\frac{ds}{dt} \quad \text{or} \quad \frac{d}{dt}\left[\tan\theta\right] = (\sec^2\theta)\frac{d\theta}{dt}.$$

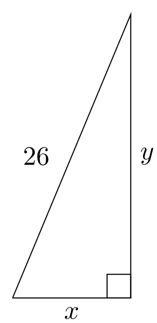
  (6) **Find the desired quantity.** Use all the known information as well as the differentiated equation from the last step to solve for the unknown quantity.

Here is an example problem: A 26 foot ladder is leaning against a vertical wall. The floor is slightly slippery so that the bottom of the ladder starts sliding away from the wall at a rate of 0.1 foot/second. At what rate is the top of the ladder sliding down the wall at the moment when the bottom of the ladder is 10 feet from the wall?

We first draw a simple picture.



The quantities that are changing are the distance between the bottom of the ladder and the wall, as well as the distance between the top of the ladder and the floor. Let's call these x and y respectively. Let's update the picture:



We know that at the given instant in time, $x = 10$ and $\frac{dx}{dt} = 0.1$. The problem is asking us to find $\frac{dy}{dt}$. The quantities are related by the Pythagorean Theorem, which says

$$x^2 + y^2 = 26^2.$$

This is our relationship between the variables in the problem. We can differentiate implicitly with respect to t to obtain

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

This is the relationship between our rates. We know x and $\frac{dx}{dt}$, and we want $\frac{dy}{dt}$. The other thing we need is y, which can be found from the above Pythagorean Theorem equation

$$y^2 = 25^2 - x^2 = 676 - 10^2 = 576.$$

This implies

$$y = \sqrt{576} = 24.$$

Plugging everything in to our equation that relates the rates,

$$2(10)(0.1) + 2(24)\frac{dy}{dt} = 0,$$

and solving for $\frac{dy}{dt}$ gives

$$\frac{dy}{dt} = -\frac{1}{24} \text{ ft/sec.}$$

Notice $\frac{dy}{dt}$ is negative, which makes sense because the top of the ladder is sliding down the wall (hence y is decreasing).

Extreme Values of Functions. Section 4.1

- We say that a function f with domain D attains an **absolute maximum** at a point $x = c$ if

$$f(c) \geq f(x) \quad \text{for all } x \text{ in } D.$$

  The value $f(c)$ is the maximum value, that is, the largest output of the function. This corresponds to the highest point on the graph. The notion of an **absolute minimum** is defined in a similar way. Functions will not always have an absolute maximum and an absolute minimum. As an example, consider $f(x) = \tan x$ on the domain $(-\pi/2, \pi/2)$. To see this, look at the graph and notice there is no largest or smallest value. The Extreme Value Theorem (stated below) provides a guarantee that these extreme values exist under certain circumstances.

- The **Extreme Value Theorem** states that a continuous function f on a closed interval $[a, b]$ always attains an absolute maximum and an absolute minimum on the domain $[a, b]$.

- We often expect that max (respectively min) values of a differentiable function f occur at points on the graph which are peaks (respectively valleys). Such points have the property that the slope of the tangent line is 0. We call the corresponding x-values **critical points** of the function f. That is,

$$c \text{ is a critical point of } f \text{ if } f'(c) = 0.$$

  Consequently, critical points provide excellent candidates for the location of the absolute max/min. The other possible location for the absolute max/min is at the endpoints a and b of the interval $[a.b]$.

- To find the absolute max/min values of a differentiable function f on the interval $[a, b]$,
  (1) Find all critical points for f in $[a, b]$. To do this, solve the equation $f'(x) = 0$ for x. The solutions are the critical points. Throw away any critical points which are not in $[a.b]$.
  (2) Evaluate your original function f at all of the critical points, as well as the endpoints a and b.

(3) The greatest output obtained in part (2) is the absolute maximum value and the least output obtained is the absolute minimum.

As an example, let's find the absolute max/min of

$$f(x) = x^3 + 3x^2 - 9x + 1 \quad \text{on } [0, 3].$$

To find the critical points, we must solve $f'(x) = 0$, which is the equation

$$3x^2 + 6x - 9 = 0.$$

Dividing by 3,

$$x^2 + 2x - 3 = 0$$

and factoring gives

$$(x - 1)(x + 3) = 0.$$

The two solutions are $x = 1$ and $x = -3$. The point -3 is not in our interval $[0, 3]$, so it means nothing for this problem. The only critical point in $[0, 3]$ is 1. Evaluating f at the critical points and the endpoints gives

$$f(1) = (1)^3 + 3(1)^2 - 9(1) + 1 = -4$$
$$f(0) = (0)^3 + 3(0)^2 - 9(0) + 1 = 1$$
$$f(3) = (3)^3 + 3(3)^2 - 9(3) + 1 = 28.$$

Thus the absolute max value is 28, and it occurs at the endpoint $x = 3$. The absolute minimum value is -4, and it occurs at the critical point $x = 1$.

For another example, let's find the absolute max/min of

$$f(x) = x^2\sqrt{5 - x} \quad \text{on } [-1, 5].$$

We first differentiate

$$f'(x) = 2x(5 - x)^{1/2} + x^2 \left(\frac{1}{2}(5 - x)^{-1/2}(0 - 1) \right),$$

that is,

$$f'(x) = 2x\sqrt{5 - x} - \frac{x^2}{2\sqrt{5 - x}}.$$

To find the critical points, we must solve $f'(x) = 0$, which is the equation

$$2x\sqrt{5 - x} - \frac{x^2}{2\sqrt{5 - x}} = 0.$$

Multiplying both sides by $2\sqrt{5 - x}$ gives

$$4x(5 - x) - x^2 = 0,$$

and so

$$20x - 5x^2 = 0.$$

Factoring,

$$5x(4 - x) = 0,$$

and so $x = 0$ and $x = 4$ are the solutions, hence our critical points. Now we evaluate f at the critical points as well at the endpoints -1 and 5:

$$\begin{aligned} f(0) &= (0)^2\sqrt{5-0} = 0 \\ f(4) &= (4)^2\sqrt{5-4} = 16 \\ f(-1) &= (-1)^2\sqrt{5-(-1)} = \sqrt{6} \\ f(5) &= (5)^2\sqrt{5-5} = 0 \end{aligned}$$

The absolute maximum value is 16, and it occurs at $x = 4$. The absolute minimum value is 0, and it occurs at both $x = 0$ and $x = 5$.

Some Important Theorems. Sections 4.2, 4.3

- **Rolle's Theorem**: If f is a continuous function on the closed interval $[a, b]$ which is differentiable on (a, b) and $f(a) = f(b)$, then there exists a point c in (a, b) for which $f'(c) = 0$.

  Rolle's Theorem basically says "what goes up must come down." It's assumed that $f(a) = f(b)$, so if, say, the function values increase as we move to the right of a, then they will eventually have to decrease in order to satisfy $f(a) = f(b)$. So there has to be a peak somewhere where the slope of the tangent line is 0. This is the point c.

- **Mean Value Theorem**: If f is a continuous function on the closed interval $[a, b]$ which is differentiable on (a, b), then there exists a point c in (a, b) for which

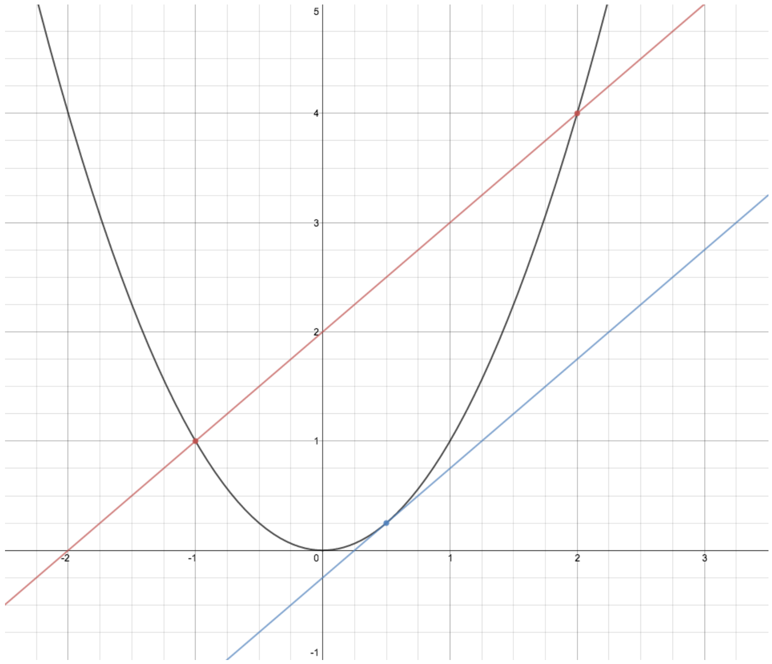
  $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

  The Mean Value Theorem says there is a point c in (a, b) at which the instantaneous rate of change of f is equal to the average (or mean) rate of change of f over the whole interval $[a, b]$. You can think of it as a "slanted" version of Rolle's Theorem. In fact, if you apply the Mean Value Theorem to a function for which $f(a) = f(b)$, then you obtain the result of Rolle's Theorem.

  In the figure, the red line is the secant line from the point where $x = -1$ to the point where $x = 2$. It's slope represents the average rate of change of f on the interval $[-1, 2]$. The blue line is the tangent line at the point $c = 1/2$. It's slope is the same as the slope of the secant line. The Mean Value Theorem guarantees that such a point c always exists.

  One physical interpretation of the Mean Value Theorem is that if you traveled from point A to point B. Then at some moment during your trip, your velocity was equal to your average velocity for the entire trip. This is the Mean Value Theorem applied to the position function $s(t)$.

- Part of the importance of the Mean Value Theorem is in its corollaries (statements which follow from it).



Mean Value Theorem for $f(x) = x^2$ on $(-1, 2)$. It turns out $c = 1/2$.

- **Corollary 1**: If $f'(x) = 0$ for all x in an interval (a, b), then $f(x) = C$ for all x in (a, b), where C is some constant.

  We previously knew that constant functions have zero derivative. This says that constant functions are the *only* functions with zero derivative.

- **Corollary 2**: If $f'(x) = g'(x)$ for all x in an interval (a, b), then there is a constant C such that $f(x) = g(x) + C$ for all x in (a, b).

  If two functions have the exact same derivative, they may not be the exact same function. However, this corollary says that this is not too far off, as they merely differ by a constant.

- **Corollary 3**: Suppose that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b).
  - ⋄ If $f'(x) > 0$ at each point x in (a, b), then f is increasing on $[a, b]$.
  - ⋄ If $f'(x) < 0$ at each point x in (a, b), then f is decreasing on $[a, b]$.

Using Derivatives to Determine the Behavior of Functions. Sections 4.3, 4.4

- As indicated by Corollary 3 above, the sign of the first derivative $f'(x)$ tells us if f is increasing or decreasing.

- The sign of the second derivative $f''(x)$ tells us about the **concavity** of the curve $y = f(x)$.

- ◇ If $f''(x) > 0$ at each point x in (a, b), then f is **concave up** on $[a, b]$. This means that the graph $y = f(x)$ is bending upwards.
- ◇ If $f''(x) < 0$ at each point x in (a, b), then f is **concave down** on $[a, b]$. This means that the graph $y = f(x)$ is bending downwards.

To understand this, you should think of f'' as $(f')'$. Think of f' as the slope of the tangent line to the graph of f, and think of the outer derivative as a rate of change. That is, $f'' = (f')'$ is the rate at which the slope of the tangent line is changing. Draw some pictures of concave up (respectively concave down) curves and convince yourself that the slope of the tangent lines is increasing (respective decreasing).

- Recall that a **critical point** of a differentiable function f is a point c at which $f'(c) = 0$. For a differentiable function f, critical points can be found by solving the equation $f'(x) = 0$ for x.
  - ◇ A critical point c for f is a *local maximum* if $f(x) \leq f(c)$ for all x near c.
  - ◇ A critical point c for f is a *local minimum* if $f(x) \geq f(c)$ for all x near c.

  There are a few ways to determine if a critical point is a local maximum, a local minimum, or neither.

- **First Derivative Test for Local Extrema:** Suppose that c is a critical point of a differentiable function f. Moving across c from left to right,
  - ◇ if $f'(x)$ changes from negative to positive at c, then f has a local minimum at c;
  - ◇ if $f'(x)$ changes from positive to negative at c, then f has a local maximum at c;
  - ◇ if $f'(x)$ does not change sign at c (that is, $f'(x)$ is positive on both sides of c or negative on both sides), then f has neither a local maximum nor a local minimum at c.

- **Second Derivative Test for Local Extrema:** Suppose c is a critical point of f, and f'' is continuous on an open interval that contains c.
  - ◇ If $f''(c) < 0$, then f has a local maximum at c.
  - ◇ If $f''(c) > 0$, then f has a local minimum at c.
  - ◇ If $f''(c) = 0$, then the test is inconclusive, meaning we can't deduce anything based on this information. Try the First Derivative Test to determine the what type of critical point c is.

- An **inflection point** for a function f is a point $x = c$ at which the concavity of the graph changes. If f is twice-differentiable, then every inflection point c must satisfy $f''(c) = 0$. So to find the inflection points, one should solve this equation for c. Notice that not every c for which $f''(c) = 0$ is an inflection point, just like how not every critical point is a local min or max.

- **Graphing:** By analyzing all the above information, we can put together pretty good sketches of functions. Here are some steps to follow to graph $y = f(x)$.

(1) Identify the domain of the function $y = f(x)$.
(2) Identify any asymptotes (vertical or horizontal).
(3) Find y' and y''.
(4) Find the critical points.
(5) Determine where the graph is increasing/decreasing.
(6) Find the points of inflection, and determine where the graph is concave up/concave down.
(7) Use the information above to classify each critical point as a local minimum, local maximum, or neither.
(8) Plot the critical points and inflection points, and sketch the curve using all the gathered information.

- **Example:** Let's sketch $y = x^4 - 2x^2$. Notice that the domain is all real numbers. So there are no vertical asymptotes. Since

$$\lim_{x\to\infty} x^4 - 2x^2 = \infty, \qquad \lim_{x\to-\infty} x^4 - 2x^2 = \infty,$$

we see there are no horizontal asymptotes. We differentiate to find

$$y' = 4x^3 - 4x, \qquad y'' = 12x^2 - 4.$$

To find the critical points, we solve $y' = 0$. So

$$\begin{aligned} 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \\ 4x(x-1)(x+1) &= 0. \end{aligned}$$

There are three critical points: $x = -1, 0, 1$.

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Evaluate $y' = 4x^3 - 4x$ somewhere in interval	$y'(-2) = -24$	$y'(-1/2) = 3/2$	$y'(1/2) = -3/2$	$y'(2) = 24$
Sign of y'	$-$	$+$	$-$	$+$
Behavior	decreasing	increasing	decreasing	increasing

Notice we can immediately use the First Derivative Test to see that there is a local minimum at -1, a local max at 0, and a local minimum at 1. To find any inflection points, we first solve $y'' = 0$. So

$$\begin{aligned} 12x^2 - 4 &= 0 \\ 3x^2 - 1 &= 0 \\ x^2 &= \frac{1}{3} \\ x &= \pm\frac{1}{\sqrt{3}}. \end{aligned}$$

We have two possible inflection points: $1/\sqrt{3}$ and $-1/\sqrt{3}$. Let's investigate the concavity.

Interval	$(-\infty, -\frac{1}{\sqrt{3}})$	$(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$	$(\frac{1}{\sqrt{3}}, \infty)$
Evaluate $y' = 12x^2 - 4$ somewhere in interval	$y''(-1) = 8$	$y''(0) = -4$	$y''(1) = 8$
Sign of y''	$+$	$-$	$+$
Behavior	concave up	concave down	concave up

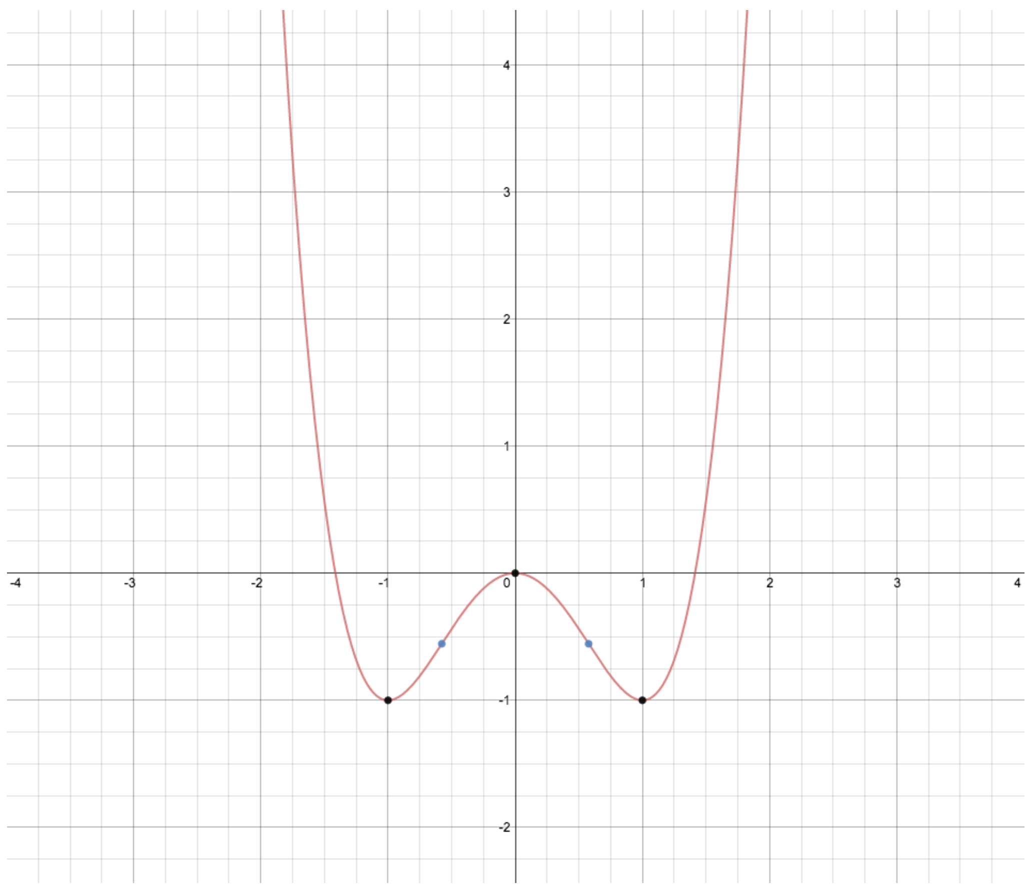
The concavity does change at both $x = 1/\sqrt{3}$ and $x = -1/\sqrt{3}$, so we have two inflection points. To plot the important points on the graph, we need to calculate the y-coordinates. For the critical points,

$$x = -1, \quad y = (-1)^4 - 2(-1)^2 = -1,$$
$$x = 0, \quad y = 0^4 - 2(0)^2 = 0,$$
$$x = 1, \quad y = 1^4 - 2(1)^2 = -1.$$

For the inflection points,

$$x = -\frac{1}{\sqrt{3}}, \quad y = \left(-\frac{1}{\sqrt{3}}\right)^4 - 2\left(-\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{9} - \frac{2}{3} = -\frac{5}{9}$$
$$x = \frac{1}{\sqrt{3}}, \quad y = \left(\frac{1}{\sqrt{3}}\right)^4 - 2\left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{9} - \frac{2}{3} = -\frac{5}{9}$$

With all this information, we can sketch the graph.



Graph of $y = x^4 - 2x^2$

Optimization Problems. Section 4.5

- In an optimization problem, the goal is the find either the maximum of minimum values of a certain function, depending on the context. The function $f(x)$ that we are maximizing/minimizing is modeled on a "real life" scenario, where we have some control over a variable x. Not only do we want the maximum/minimum values of f, but we often want to know what choice of input x must we make to get these optimal values.

- The calculus techniques needed here are not new (see section 4.1). Once we have the function $f(x)$, we find its min/max values by looking for critical points. The critical points are found by solving the equation $f'(x) = 0$.

- The difficulty in these problems is often correctly modeling the function $f(x)$ given the "real life" scenario. It's not really possible to describe a general method for how to do this. One piece of advice is to think in terms of simple formulas, expressed in words, for example
$$\text{distance} = (\text{speed})(\text{time})$$
$$\text{volume of box} = (\text{length})(\text{width})(\text{height})$$
$$\text{profit} = \text{revenue} - \text{cost}$$
$$\text{total cost} = (\#\text{ of units})(\text{cost per unit})$$
Doing this helps you to break down the problem to simpler pieces. The individual components of these equations can usually be expressed in terms of the variable x that you have control over in the problem.

  Another issue which arises is that sometimes it appears as if there are two variables that you have control over. For example, you may be designing a cylindrical can whose dimensions can be described by the height h of the can and the radius r of the base cylinder. In situations like these, you should look for some kind of constraint that gives a relationship between the two variables. For example, if the problem says that the can must have a volume of $216\,\text{in}^3$, then we get the equation
$$\pi r^2 h = 216.$$
From this, we can solve for h in terms of r:
$$h = \frac{216}{\pi r^2}.$$
This effectively eliminates the second variable h, since we can replace any future occurrences of h with $\frac{216}{\pi r^2}$. Ultimately, everything should be able to be expressed as a function of the variable r. Then we can differentiate with respect to r, find critical points, etc.

- **Example:** You are designing a rectangular poster to contain $50\,\text{in}^2$ of printing with a $4\,\text{in}$ margin at the top and bottom and a $2\,\text{in}$ margin at each side. What overall dimensions will minimize the amount of paper used?

  First, you should draw a picture (do it). Let's let w denote the width of the poster (from left to right) and let h denote the height of the poster (from top to bottom). We would like to minimize the amount of paper used, which is proportional to the area of the poster. So we should minimize the area $A = wh$. The problem (as discussed above) is that there are two parameters w and h, and we can't apply our methods until we have just a single variable. There is some kind of implied relationship between w and h, because the poster is supposed to contain $50\,\text{in}^2$ of printing. So

$$\begin{aligned}50 &= \text{Area of printing}\\ &= (\text{Width of printed region})(\text{Height of printed region})\\ &= (w-4)(h-8).\end{aligned}$$

Notice that the width/height of the printed region was obtained by taking the width/height of the posted and subtracting the margins. So

$$50 = (w-4)(h-8) = wh - 8w - 4h + 32.$$

We can solve for one of the variables, say let's solve for w. Subtracting 32, we have

$$\begin{aligned}wh - 8w - 4h &= 18\\ w(h-8) &= 18 + 4h\\ w &= \frac{18+4h}{h-8}.\end{aligned}$$

Now we return to our area function $A = wh$, which we want to maximize. By substitution, we can express A as a function of h:

$$A(h) = \frac{18+4h}{h-8}h = \frac{18h+4h^2}{h-8}.$$

This is what we need to minimize. By the quotient rule,

$$A'(h) = \frac{(18+8h)(h-8) - (18h+4h^2)(1)}{(h-8)^2} = \frac{4h^2 - 64h - 144}{(h-8)^2}.$$

Setting $A'(h) = 0$, we must solve

$$\begin{aligned}\frac{4h^2-64h-144}{(h-8)^2} &= 0\\ 4h^2 - 64h - 144 &= 0\\ h^2 - 16h - 36 &= 0\\ (h+2)(h-18) &= 0.\end{aligned}$$

So $h = -2$ or $h = 18$. Since $h = -2$ makes no physical sense, $h = 18$ is what we are looking for. The other dimension can be found from h,

$$w = \frac{18+4(18)}{18-8} = \frac{90}{10} = 9.$$

So the dimensions of the poster should be 9×18.

Riemann Sums and Integration. Sections 5.1, 5.2, 5.3

- Consider the problem of calculating the area under the curve $y = f(x)$, above the x-axis, and between $x = a$ and $x = b$. This area can be *estimated* by a **Riemann sum**. To do this, we choose a number n and partition the interval $[a, b]$ into n equal sized subintervals, each of which has width $\Delta x = (b-a)/n$. Label the endpoints of these intervals as $x_0, x_1, x_2, \ldots, x_n$, so that $[x_{k-1}, x_k]$ is the k-th subinterval. In each subinterval $[x_{k-1}, x_k]$, we choose a sample point which we call x_k^*. Above each subinterval, we draw a rectangle whose height is determined by the function value at the sample point. That is, the height is $f(x_k^*)$. Collectively, these rectangles serve as an approximation to the region under the curve. We can approximate the area under the curve by calculating the area in the rectangles. The area of any rectangle is its width times its height. The width of all of our rectangles is Δx, and the height of the k-th rectangle is $f(x_k^*)$. Thus the sum of the area in the rectangles is the

$$\text{Riemann sum:} \quad \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

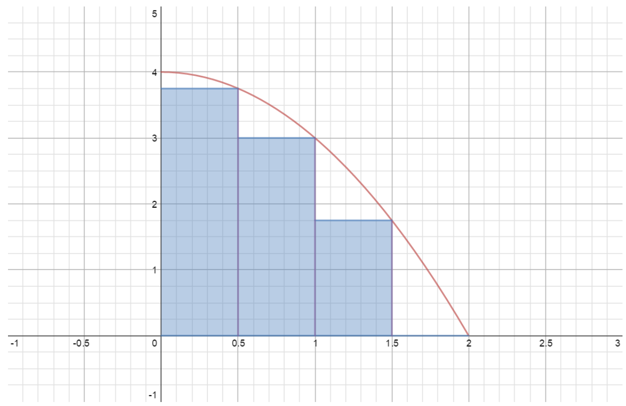
- For the sample points x_k^*, any point can be chosen. However, it is usually reasonable to make some kind of uniform choice. For example, one can always use the left endpoint x_{k-1} of the interval $[x_{k-1}, x_k]$ as the sample point x_k^*. An alternative choice would be to use the right endpoint. A third option would be to take the midpoint of the interval $[x_{k-1}, x_k]$ as the sample point.

- The estimate provided by the Riemann sum gets better and better the larger n gets. By taking some kind of limit as $n \to \infty$, we expect to get the exact amount of area under the curve. The **integral** of $f(x)$ on $[a, b]$ is defined to be this limit:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

  Notationally, the integral is supposed to resemble a Riemann sum. The integral sign $\int$ is an elongated "S" for "summation", much like the $\sum$ represents a summation in the Riemann sum. Integrals can be thought of as a sort of "continuous sum," as opposed to the finite Riemann sum (there's only finitely many terms there). The dx appearing in the integral could be thought of as what happens to the Δx as $n \to \infty$. It shrinks to the "infinitesimal" length dx. So $f(x)dx$ can be thought of as the area of an infinitesimally thin rectangle with height $f(x)$. Putting it all together, the integral $\int_a^b f(x)dx$ is a "continuous sum" of the areas $f(x)dx$ of all the "infinitesimally thin" rectangles under the curve, between $x = a$ and $x = b$. The total sum is exactly the area under the curve, as desired.

- The integral $\int_a^b f(x)dx$ is the *exact* area under the curve $y = f(x)$ between $x = a$ and $x = b$. If $f(x) < 0$, so that the curve is actually *below* the x-axis, then the integral calculates the negative of the area between the curve and



$(n = 4)$ Riemann sum $= 4.25$

the x-axis.

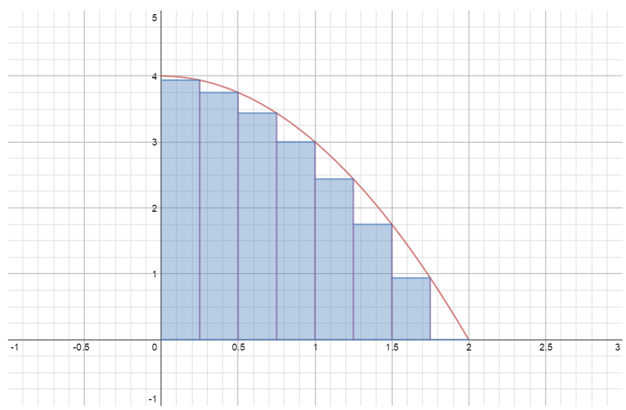
- As an example, consider the problem of calculating the area under the curve $y = f(x) = 4 - x^2$, above the x-axis, and between $x = 0$ and $x = 2$. Let's first take $n = 4$, that is, let's do a Riemann sum with 4 rectangles. We'll use the right endpoints of the intervals as our sample points. The length of the intervals will be $2/4 = 1/2$. The Riemann sum is

$$\begin{aligned}
\text{Riemann Sum} &= \frac{1}{2}f\left(\frac{1}{2}\right) + \frac{1}{2}f(1) + \frac{1}{2}f\left(\frac{3}{2}\right) + \frac{1}{2}f(2) \\
&= \frac{1}{2}\left(4 - (1/2)^2\right) + \frac{1}{2}(4 - 1^2) + \frac{1}{2}(4 - (3/2)^2) + \frac{1}{2}(4 - 2^2) \\
&= \frac{1}{2}\left(\frac{15}{4}\right) + \frac{1}{2}(3) + \frac{1}{2}\left(\frac{7}{4}\right) + \frac{1}{2}(0) \\
&= \frac{15 + 12 + 7 + 0}{8} \\
&= \frac{17}{4}.
\end{aligned}$$

A picture of our Riemann Sum is shown on the next page. Notice the fourth rectangle actually has height 0, so it looks like its not there.

We can do better by trying a Riemann sum with more rectangles, say $n = 8$. We'll still use right endpoints as our sample points. Here the interval length is $2/8 = 1/4$, and so the Riemann sum is

$$\begin{aligned}\text{Riemann Sum } &= \frac{1}{4}f\left(\frac{1}{4}\right) + \frac{1}{4}f\left(\frac{1}{2}\right) + \frac{1}{4}f\left(\frac{3}{4}\right) + \frac{1}{4}f(1) \\ &\qquad + \frac{1}{4}f\left(\frac{5}{4}\right) + \frac{1}{4}f\left(\frac{3}{2}\right) + \frac{1}{4}f\left(\frac{7}{4}\right) + \frac{1}{4}f(2) \\ &= \frac{1}{4}(4-(1/4)^2) + \frac{1}{4}(4-(1/2)^2) + \frac{1}{4}(4-(3/4)^2) + \frac{1}{4}(4-1^2) \\ &\qquad + \frac{1}{4}(4-(5/4)^2) + \frac{1}{4}(4-(3/2)^2) + \frac{1}{4}(4-(7/4)^2) + \frac{1}{4}(4-2^2) \\ &= (\text{arithmetic}) \\ &= \frac{77}{16}.\end{aligned}$$

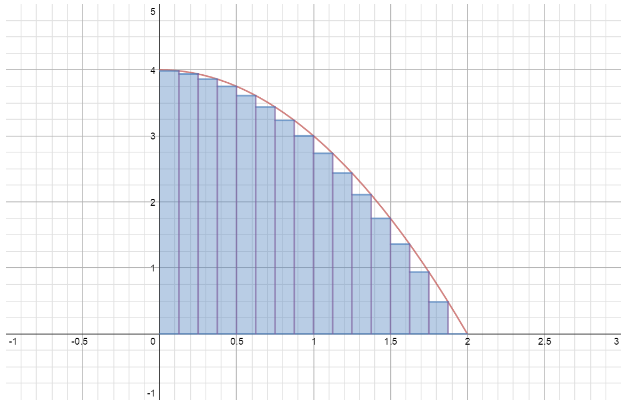


$(n = 8)$ Riemann sum $= 4.8125$

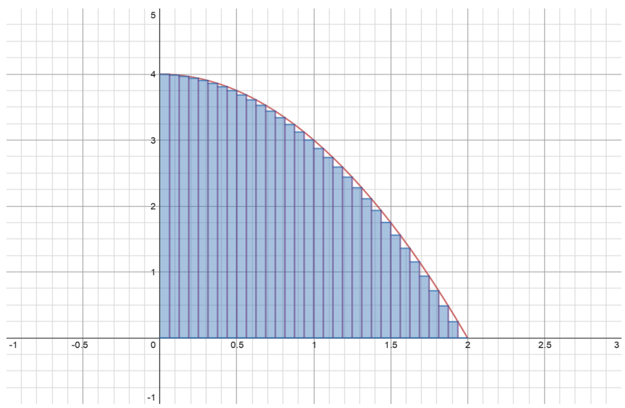
To get better estimates, we can raise n and repeat the process. Observe the pictures for $n = 16, 32, 50$, as well as the corresponding values of the Riemann sums. Notice how as n gets larger, the union of the all the rectangles looks more and more like the curved region. The value of the Riemann sum gets closer to the actual area, which is given by the integral $\int_0^2 (4-x^2)dx$. Using the Fundamental Theorem of Calculus (see below), the integral is

$$\begin{aligned}\int_0^2 (4-x^2)dx &= \left[4x - \frac{x^3}{3}\right]_0^2 \\ &= \left(4(2) - \frac{2^3}{3}\right) - \left(4(0) - \frac{0^3}{3}\right) \\ &= 8 - \frac{8}{3} \\ &= \frac{16}{3} = 5.\overline{3}.\end{aligned}$$

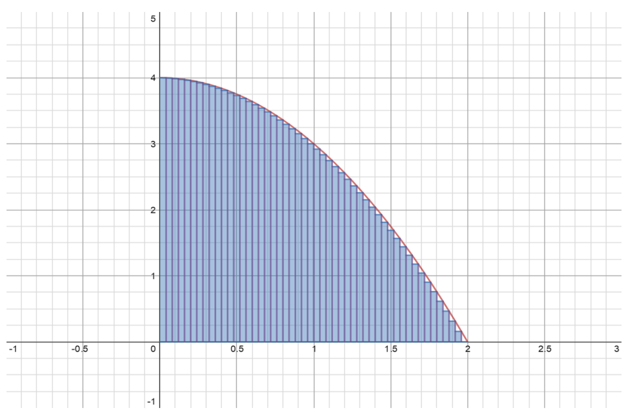
Notice the Riemann sums are getting closer to the exact value. If you are less than impressed with the amount of error we have, it is because we are



$(n = 16)$ Riemann sum $= 5.078125$



$(n = 32)$ Riemann sum $= 5.20703125$



$(n = 50)$ Riemann sum $= 5.2528$

being somewhat lazy and using rectangles which are flat on top. There are more sophisticated techniques to obtain better results. If you are interested, look up the Trapezoidal Rule or Simpson's Rule on Wikipedia. Since computers can do arithmetic extremely quickly, one could always program a computer to perform a Riemann sum for a very large n to get a very good approximation of an integral. So if you encounter an integral that you can't compute exactly (this does happen), you can always, in theory, approximate it to whatever precision you need by using these numerical methods.

- Definite integrals have the following properties.
  - ◇ $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx.$
  - ◇ $\displaystyle\int_a^a f(x)dx = 0.$
  - ◇ $\displaystyle\int_a^b kf(x)dx = k\int_a^b f(x)dx$ for any constant k.
  - ◇ $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$
  - ◇ $\displaystyle\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx.$
- Definite integrals are defined in a quite complicated way, as a limit of Riemann sums. One might worry that this limit might not exist. The good news is that for most reasonable functions you would encounter, the limit will exist, and so the integral makes sense. If the limit defining $\int_a^b f(x)dx$ exists, then we say $f(x)$ is **integrable** on the interval $[a, b]$. It is a fact that every function $f(x)$ which is continuous on $[a, b]$ will also be integrable on $[a, b]$. So we can always integrate continuous functions.

Antiderivatives. Section 4.7

- An **antiderivative** of a function $f(x)$ is another function $F(x)$ with the property that $F'(x) = f(x)$. For example, $3x^2$ is an antiderivative of $6x$, because
$$\frac{d}{dx}\left[3x^2\right] = 6x.$$
Notice that $3x^2 - 17$ is also an antiderivative of $6x$. In fact anything of the form $3x^2 + C$, where C is a constant, is an antiderivative of $6x$. Moreover, Corollary 2 of the Mean Value Theorem implies that every antiderivative $F(x)$ of $6x$ is of the form $F(x) = 3x^2 + C$ for some constant C. Indeed, this follows because $F(x)$ and $3x^2$ have the same derivative, so they must differ by a constant. Here we would say that $3x^2 + C$ is the **general antiderivative** of $6x$, because it describes *all* antiderivatives of $6x$.

  There is nothing special about the example $6x$. Let $f(x)$ be any continuous function. For the reasons described above, any two antiderivatives $F(x)$ and $G(x)$ of $f(x)$ must satisfy $F(x) = G(x) + C$ for some constant C. So if we can find a single antiderivative, we can describe the general antiderivative.

- **Power Rule (for antiderivatives):** If $n \neq -1$, then the general antiderivative of x^n is $\dfrac{x^{n+1}}{n+1} + C$.
  - ◇ The general antiderivative of x^3 is $\frac{x^4}{4} + C$.
  - ◇ The general antiderivative of x^{16} is $\frac{x^{17}}{17} + C$.
  - ◇ To find the general antiderivative of $\frac{1}{x^3}$, notice that $\frac{1}{x^3} = x^{-3}$. The power rule says that the general antiderivative is $\frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$.
  - ◇ A similar trick works for radicals. Notice that $\sqrt{x} = x^{1/2}$. So the general antiderivative of $\sqrt{x}$ is $\frac{x^{3/2}}{3/2} + C = \frac{2}{3}x^{3/2} + C$.

- Every differentiation rule has a corresponding antidifferentiation rule. So all the derivatives of trigonometric functions tell us certain antiderivatives:
  - ◇ The general antiderivative of $\cos x$ is
  $$\sin x + C.$$
  - ◇ The general antiderivative of $\sin x$ is
  $$-\cos x + C.$$
  - ◇ The general antiderivative of $\sec^2 x$ is
  $$\tan x + C.$$
  - ◇ The general antiderivative of $\csc^2 x$ is
  $$-\cot x + C.$$
  - ◇ The general antiderivative of $\sec x \tan x$ is
  $$\sec x + C.$$
  - ◇ The general antiderivative of $\csc x \cot x$ is
  $$-\csc x + C.$$

- Just like derivatives, we can calculate antiderivatives term by term. Multiplicative constants can be treated just as they are when doing derivatives. For example, the general antiderivative of
$$f(x) = 3x^4 + 7x - 8 + \sin x$$
is
$$F(x) = 3\left(\frac{x^5}{5}\right) + 7\left(\frac{x^2}{2}\right) - 8\left(\frac{x^1}{1}\right) + (-\cos x) + C,$$
which simplifies to
$$F(x) = \frac{3}{5}x^5 + \frac{7}{2}x^2 - 8x - \cos x + C.$$

The Fundamental Theorem of Calculus. Section 5.4

- **Fundamental Theorem of Calculus (Part 1):** Let $f(x)$ be a continuous function and let a be a number. Then the function

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of $f(x)$. In other words,

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

- You can think of this as a new differentiation rule. The Fundamental Theorem of Calculus tells us how to differentiate functions where the variable x appears as a limit of integration. Notice that in the function $F(x)$ above, the variable t is a dummy variable used to define the integral. The independent variable of the function F is x. By combining this with the chain rule, we obtain

$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x))g'(x).$$

Some examples:
  - ◇ $\dfrac{d}{dx}\displaystyle\int_{-17}^x \frac{t^2+1}{t^2+t+1}dt = \frac{x^2+1}{x^2+x+1}.$
  - ◇ $\dfrac{d}{dx}\displaystyle\int_x^8 \sqrt{t}\sin t\ dt = \frac{d}{dx}\left[-\int_8^x \sqrt{t}\sin t\ dt\right] = -\sqrt{x}\sin x.$
  - ◇ $\dfrac{d}{dx}\displaystyle\int_1^{x^2+4x} \frac{1}{t}\ dt = \frac{1}{x^2+4x}(2x+4) = \frac{2x+4}{x^2+4x}.$

- **Fundamental Theorem of Calculus (Part 2):** Let $F(x)$ be any antiderivative of the function $f(x)$. Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

- This part of the Fundamental Theorem of Calculus is the big theorem that tells us how to actually *compute* an integral. To compute an integral, we first find an antiderivative $F(x)$ of the integrand $f(x)$. Then we simply evaluate F at the endpoints b and a, and subtract. We use the notation

$$F(x)\Big]_a^b = F(b) - F(a).$$

So a typical integral computation looks like

$$\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a).$$

The issue of finding an antiderivative is what makes integration difficult. However, there are many instances where we can find antiderivatives relatively easily (see above). Here are some examples.

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \cos x \; dx &= \sin x \Big]_{-\pi/2}^{\pi/2} \\ &= \sin(\pi/2) - \sin(-\pi/2) \\ &= 1 - (-1) = 2.\end{aligned}$$

$$\begin{aligned}\int_{-1}^{2} (8x^3 - 10x + 8)dx &= \left[8\left(\frac{x^4}{4}\right) - 10\left(\frac{x^2}{2}\right) + 8x\right]_{-1}^{2} \\ &= \left[2x^4 - 5x^2 + 8x\right]_{-1}^{2} \\ &= \big(2(2)^4 - 5(2)^2 + 8(2)\big) - \big(2(-1)^4 - 5(-1)^2 + 8(-1)\big) \\ &= (32 - 20 + 16) - (2 - 5 - 8) \\ &= 28 - (-11) \\ &= 39.\end{aligned}$$

Sometimes it helps to simplify the integrand first.

$$\begin{aligned}\int_{1}^{2} \frac{2x^4 + x^3 - 8}{x^3} dx &= \int_{1}^{2} \left(\frac{2x^4}{x^3} + \frac{x^3}{x^3} - \frac{8}{x^3}\right) dx \\ &= \int_{1}^{2} \left(2x + 1 - 8x^{-3}\right) dx \\ &= \left[2\left(\frac{x^2}{2}\right) + x - 8\left(\frac{x^{-2}}{-2}\right)\right]_{1}^{2} \\ &= \left[x^2 + x + \frac{4}{x^2}\right]_{1}^{2} \\ &= \left(2^2 + 2 + \frac{4}{2^2}\right) - \left(1^2 + 1 + \frac{4}{1^2}\right) \\ &= (4 + 2 + 1) - (1 + 1 + 4) \\ &= 1.\end{aligned}$$

- The Fundamental Theorem of Calculus can be interpreted as saying that differentiation and integration are inverse operations, once this is suitably interpreted. Part 1 says that if you take a function $f(x)$, form the integral function $\int_a^x f(t)dt$, and then differentiate it with respect to x, you obtain the function $f(x)$ that you started with:

  $$\frac{d}{dx} \int_a^x f(t)dt = f(x).$$

  So in this sense, the derivative of the integral is the original integrand. Rewritten slightly, Part 2 says that

  $$\int_a^b f'(x)dx = f(b) - f(a).$$

This says the integral of the derivative of $f(x)$ can be simply expressed in terms of the function $f(x)$ itself. One way to come to grips with this, and really understand it, is to notice that the quantity $f(b)-f(a)$ is the change in the output of the function f between $x=a$ and $x=b$. On the other hand, let's look at the integral $\int_a^b f'(x)dx$. As discussed above, this is a sort of "continuous sum" of the values $f'(x)dx$. The continuous sum is the limit of a finite Riemann sum, where we would consider intervals $[x, x+\Delta x]$ of length Δx. By linearization, the quantity $f'(x)\Delta x$ approximates the change in the values of f over this interval:

$$f'(x)\Delta x \approx f(x+\Delta x) - f(x).$$

So adding up all the changes in f over all the subintervals gives an approximation to $f(b)-f(a)$. That is,

$$\text{Riemann sum} \ \approx \ f(b) - f(a).$$

The Fundamental Theorem of Calculus says that this becomes perfect equality when we take the limit of the Riemann sums:

$$\int_a^b f'(x)dx = f(b) - f(a).$$

It is instructive to think of the physical example where the function was $s(t)$, the position of some object at time t. Then the derivative $s'(t)$ is the velocity, and by integrating the velocity

$$\int_a^b s'(t)dt = s(b) - s(a),$$

we obtain $s(b)-s(a)$, which represents the (net) distance traveled between times $t=a$ and $t=b$. For a Riemann sum which approximates this integral, we add up expressions of the form $s'(t)\Delta t$. This quantity represents an approximation of the distance traveled over the short period of time Δt. Adding them all up gives an approximation to the total net distance traveled, as predicted by the Fundamental Theorem of Calculus.

- Now that we understand the Fundamental Theorem of Calculus, we see that computing integrals is all about finding an antiderivative for the integrand. This is often a difficult problem and deserves much attention (It's much more difficult, in general, to compute antiderivatives than it is to compute derivatives.) The following is a notational convenience. We define the **indefinite integral** $\displaystyle\int f(x)dx$ to be the general antiderivative of $f(x)$, that is, the set of all antiderivatives of $f(x)$. For example,

$$\int x^2 dx = \frac{x^3}{3} + C.$$

Notationally, an indefinite integral is written without limits of integration (as opposed to a definite integral, which has limits of integration). Practically speaking, computing an indefinite integral is the same as the first step in computing a definite integral: you find an antiderivative of the integrand. Notice you don't need limits of integration to do this. Indefinite integrals are convenient to express antiderivatives.

- The **Power Rule for Integrals** says that if $n \neq -1$, then

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

- The derivatives of the six trigonometric functions imply the following.

$$\int \cos x \; dx = \sin x + C \qquad \int \sin x \; dx = -\cos x + C$$
$$\int \sec^2 x \; dx = \tan x + C \qquad \int \csc^2 x \; dx = -\cot x + C$$
$$\int \sec x \tan x \; dx = \sec x + C \qquad \int \csc x \cot x \; dx = -\csc x + C$$

The Substitution Rule for Integrals. Sections 5.5, 5.6

- At this point, we can only do integrals that can be solved using the Power Rule or the six trigonometric integrals just listed. For more complicated integrals, we need a new technique to reduce such complicated integrals into the easy ones above that we know how to handle. The **Substitution Rule**

$$\int f(g(x))g'(x)dx = \int f(u)du$$

is a tool to accomplish this. Every differentiation rule has a corresponding integration (antidifferentiation) rule. The Substitution Rule is the integration rule corresponding to the Chain Rule in differentiation. This equality should be read from left to right. That is, the left side represents the complicated integral we start with, and we would like to use this rule to turn it into the simpler right hand side. What the rule says is that we can choose some complicated expression $g(x)$ in our integral and simply replace it with $u = g(x)$, *provided* that we can also substitute the differential $du = g'(x)dx$ into the integral.

- Here's how it looks in practice. Consider the complicated integral

$$\int \frac{9x^2}{\sqrt{1-x^3}} dx.$$

Messy expressions underneath square roots are often good candidates for substitutions. Let's try the substitution

$$u = 1 - x^3$$
$$du = -3x^2 dx.$$

To make this substitution, we will need to pair a $-3x^2$ with the dx in order to bring du in:

$$\int \frac{9x^2}{\sqrt{1-x^3}} dx = \int \frac{-3}{\sqrt{1-x^3}}(-3x^2 dx) = \int \frac{-3}{\sqrt{u}} du.$$

We've successfully performed the substitution[1] and the good news is that the integral we've gotten is much easier than the one we started with. This one we can compute with the Power Rule:

$$\begin{aligned}\int \frac{9x^2}{\sqrt{1-x^3}}dx &= \frac{-3}{\sqrt{u}}du \\ &= \int -3u^{-1/2}du \\ &= \frac{-3u^{1/2}}{1/2} + C \\ &= -6\sqrt{u} + C \\ &= -6\sqrt{1-x^3} + C.\end{aligned}$$

For an indefinite integral, it's important to convert back to the original variable x after the integration is performed. The problem asked for an antiderivative of $\frac{9x^2}{\sqrt{1-x^3}}$, so your answer should be presented in terms of x. After doing a problem like this, you should get in the habit of checking that your answer is correct by differentiating it and verifying that the result is the original integrand. Go ahead and do it for $-6\sqrt{1-x^3}$.

- Consider the example

$$\int \frac{x^2+1}{(x-1)^4}dx.$$

A good place to start is to try to get the $(x-1)^4$ under control. We can try

$$\begin{aligned}u &= x - 1 \\ du &= 1 \cdot dx = dx.\end{aligned}$$

This shouldn't be too bad because $du = dx$. However, we have to confront that extra x^2+1 on top. Remember all the x's must turn into u's. Here

[1]All of the x's including the dx have to be written in terms of the new variable u. You cannot integrate some kind of "hybrid" integral written in terms of both x and u. It just doesn't work. If you can't seem to convert all the x's to u's, then it's possible you tried a substitution that's not going to work. Sometimes trial and error is required. Try something else instead.

we can solve $u = x - 1$ for x to get $x = u + 1$. Then

$$\begin{aligned}
\int \frac{x^2+1}{(x-1)^4}dx &= \int \frac{(u+1)^2+1}{u^4}du \\
&= \int \frac{u^2+2u+2}{u^4}du \\
&= \int \left(\frac{u^2}{u^4} + \frac{2u}{u^4} + \frac{2}{u^4}\right) du \\
&= \int \left(u^{-2} + 2u^{-3} + 2u^{-4}\right) du \\
&= \frac{u^{-1}}{-1} + \frac{2u^{-2}}{-2} + \frac{2u^{-3}}{-3} + C \\
&= -\frac{1}{u} - \frac{1}{u^2} - \frac{2}{3u^3} + C \\
&= -\frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{2}{3(x-1)^3} + C.
\end{aligned}$$

This answer is also equal to

$$\frac{-x^2+x-\frac{2}{3}}{(x-1)^3} + C$$

if one goes through the effort of combining the terms together. The latter is easier to differentiate if you want to check your answer.

- For a definite integral, the Substitution Rule looks like

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Here, when we perform the substitution, we should also convert to limits of integration from the given x-values to their corresponding u-values. The upside is that we won't have to convert everything back to x at the end of the problem. Consider the example

$$\int_0^{1/4} \sec^2(\pi x)\tan^2(\pi x)dx.$$

First, convince yourself that trying $u = \sec(\pi x)$ isn't a great idea because of what du is. It turns out a better idea is to try

$$\begin{aligned}
u &= \tan(\pi x) \\
du &= \pi \sec^2(\pi x).
\end{aligned}$$

We will also have to convert the limits 0 and 1/4, which are x-values, to the corresponding u values. To do this, we use $u = \tan(\pi x)$.

$$\begin{aligned}
\text{Lower limit: } x = 0 \quad &\Rightarrow \quad u = \tan(\pi(0)) = \tan(0) = 0. \\
\text{Upper limit: } x = 1/4 \quad &\Rightarrow \quad u = \tan(\pi(1/4)) = \tan(\pi/4) = 1.
\end{aligned}$$

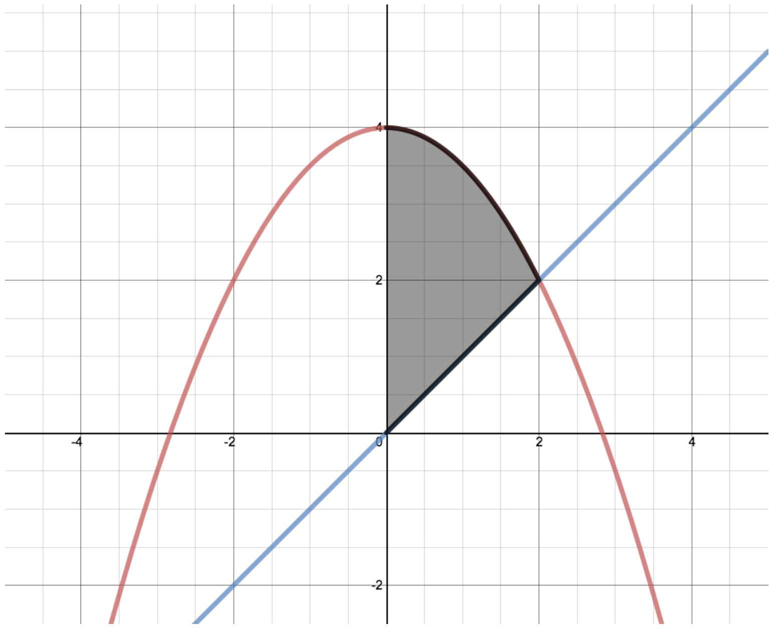
Now for the substitution. Notice we can introduce an extra factor of π as long as we also divide by π:

$$\begin{aligned}\int_0^{1/4} \sec^2(\pi x)\tan^2(\pi x)dx &= \int_0^{1/4} \frac{1}{\pi}\tan^2(\pi x)\left(\pi\sec^2(\pi x)dx\right)\\ &= \int_0^1 \frac{u^2}{\pi}du\\ &= \frac{u^3}{3\pi}\bigg]_0^1\\ &= \frac{1^3}{3\pi} - \frac{0^3}{3\pi} = \frac{1}{3\pi}.\end{aligned}$$

Areas between Curves. Section 5.6

- We know that for a positive function $f(x)$, the integral
$$\int_a^b f(x)dx$$
is the area underneath the curve $y = f(x)$ and above the x-axis between $x = a$ and $x = b$. What if we want the area below $y = f(x)$ and above another curve $y = f(x)$? The general principle is that the area below $y = f(x)$ and above $y = g(x)$ between $x = a$ and $x = b$ is given by the integral
$$\int_a^b \big[f(x) - g(x)\big]dx.$$
Notice if $g(x) = 0$, the constant function whose graph is the x-axis, then we recover the usual formula for the area under the curve $y = f(x)$ and above the x-axis as the integral of $f(x)$.

- As an example, let's find the area of the region to the right of the y-axis which is bounded by the parabola $y = 4 - \frac{x^2}{2}$ and the line $y = x$. To begin, one should always draw a picture in order to see the region in question. Notice that $y = 4 - \frac{x^2}{2}$ is a downward-facing parabolar whose maximum occurs at the point $(0, 4)$. From the picture, it's clear that we want the area below the parabola and above the line. It's clear that the lower limit of integration will be $x = 0$, but we need to find the point of intersection of the graphs to determine the upper limit. To do this, we set our two functions ($y = x$ and $y = 4 - \frac{x^2}{2}$) equal to each other, and solve for x:
$$\begin{aligned} x &= 4 - \frac{x^2}{2}\\ \frac{x^2}{2} + x - 4 &= 0\\ x^2 + 2x - 8 &= 0\\ (x-2)(x+4) &= 0\\ x &= 2, -4.\end{aligned}$$



Clearly, $x = 2$ is the one we are looking for. The area we want is

$$\begin{aligned}\text{Area} &= \int_0^2 \left[\left(4 - \frac{x^2}{2}\right) - x\right] dx \\ &= \left[4x - \frac{x^3}{6} - \frac{x^2}{2}\right]_0^2 \\ &= (4(2) - \frac{2^3}{6} - \frac{2^2}{2}) - (4(0) - \frac{0^3}{6} - \frac{0^2}{2}) \\ &= 8 - \frac{4}{3} - 2 = \frac{14}{3}.\end{aligned}$$

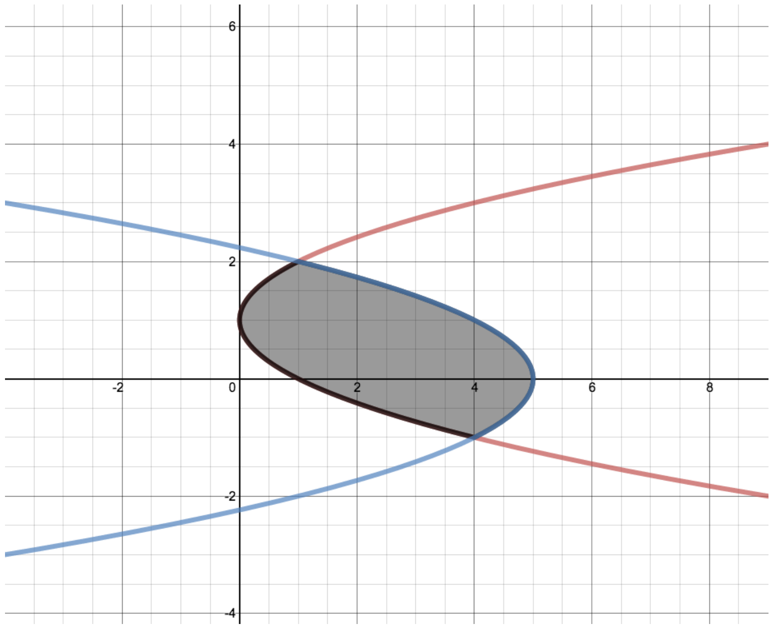
- Sometimes it is useful to integrate with respect to y instead. Suppose we want to find the area enclosed by the two parabolas

$$x = (y-1)^2 \quad \text{and} \quad x = 5 - y^2.$$

The first parabola is a vertical shift of the standard rightward-opening parabola, and the second is a horizontal shift of the standard leftward-opening parabola. We can sketch a picture:

This looks quite awkward to try to describe as an area below a certain curve and above another curve, because the curves which it is above/below change for different choices of x. Instead, this is more naturally described as a region to the left of $x = 5 - y^2$ and to the right of $x = (y-1)^2$. The area should be given by an integral of the form

$$\int \left[(5 - y^2) - (y-1)^2\right] dy,$$



but we need to find the points of intersection to find the limits of integration. To do this, we solve

$$\begin{aligned}(y-1)^2 &= 5-y^2\\ y^2-2y+1 &= 5-y^2\\ 2y^2-2y-4 &= 0\\ y^2-y-2 &= 0\\ (y-2)(y+1) &= 0\\ y &= 2,-1.\end{aligned}$$

These are our limits of integration. So

$$\begin{aligned}
\text{Area} &= \int_{-1}^{2} \big[(5-y^2)-(y-1)^2\big]\,dy \\
&= \int_{-1}^{2} \big[(5-y^2)-(y^2-2y+1)\big]\,dy \\
&= \int_{-1}^{2} \big[-2y^2+2y+4\big]\,dy \\
&= \Big[-\frac{2y^3}{3}+y^2+2y\Big]_{-1}^{2} \\
&= \left(-\frac{2(2)^3}{3}+2^2+2(2)\right)-\left(-\frac{2(-1)^3}{3}+(-1)^2+2(-1)\right) \\
&= \left(-\frac{16}{3}+4+4\right)-\left(\frac{2}{3}+1-2\right) \\
&= -\frac{16}{3}+8-\frac{2}{3}+1 \\
&= -\frac{18}{3}+9 \\
&= -6+9=3.
\end{aligned}$$

Volumes of Solids. Sections 6.1, 6.2

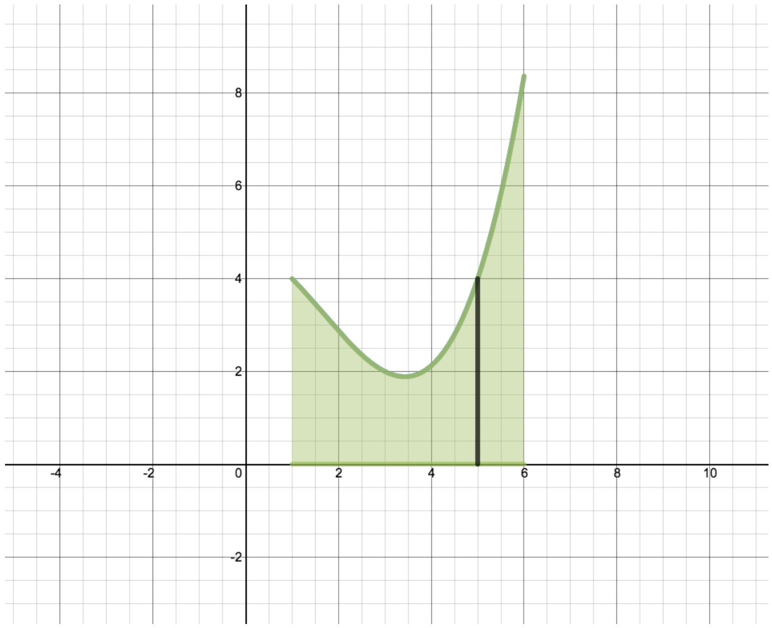
- Integration can be used to calculate the volumes of solid objects. The key idea is to consider the **cross-sections** of the solid object. These are the planar regions obtained by slicing the solid with a plane. The volume of the solid can be reconstructed by integrating all of the cross-sectional areas. Consider a solid object between $x = a$ and $x = b$. Let $A(x)$ denote the area of the cross-section of the solid with the plane perpendicular to the x-axis at the point x. At different x's, we have different cross-sections, so the cross-sectional area is a function of x. Then the volume of the solid is

$$\text{Volume } = \int_a^b A(x)dx.$$

- Our main focus will be on **solids of revolution**. These are solids which are obtained by rotating a planar region about an axis. For such a solid, the cross-sections by planes perpendicular to the axis of rotation are circles (or can be described simply in terms of circles – see the washer method.)

- **The Disk Method:** Suppose we want to calculate the volume obtained by rotating the following region about the x-axis.

  The dark line represents a generic value of x. At that x, the line segment will trace out a disk when rotated about the x-axis. This is our cross-section. To compute its area, we use the formula for the area of a circle. Here the radius $R(x)$ depends on x, and is given by the length of the line



segment. Thus the cross-sectional area is $A(x) = \pi\big[R(x)\big]^2$. The volume is

$$\text{Volume } = \int_1^6 \pi\big[R(x)\big]^2 dx.$$

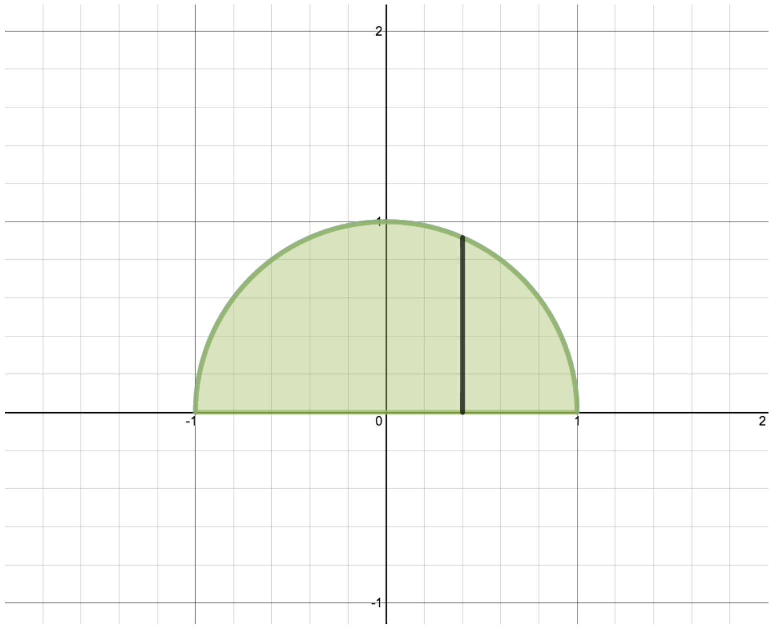
The formula for $R(x)$ will be the formula for the function whose graph forms the upper boundary of our region. In a problem, this can be explicitly determined.

- As an example, let's derive the formula for the volume of a sphere of radius r. The sphere of radius r can be obtained by rotating the circle $x^2+y^2 = r^2$ of radius r about the x-axis. Actually, we only need the top half of the circle, which we can write as a function of x by solving for y:

$$\begin{aligned} x^2 + y^2 &= r^2 \\ y^2 &= r^2 - x^2 \\ y &= \pm\sqrt{r^2 - x^2}. \end{aligned}$$

This is actually two functions: $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$. The first is the top half of the circle and the second is the bottom half of the circle. So the sphere of radius r is obtained by rotating the region below $y = \sqrt{r^2 - x^2}$ about the x-axis. Below is a picture (for the $r = 1$ case).

The line segment represents the part of the region above a generic value of x. When this segment is rotated about the x-axis, it forms a cross-sectional disk whose radius is $R(x) = \sqrt{r^2 - x^2}$, the heigh of the semicircle at that x. We can get the volume by integrating the areas of these disks as

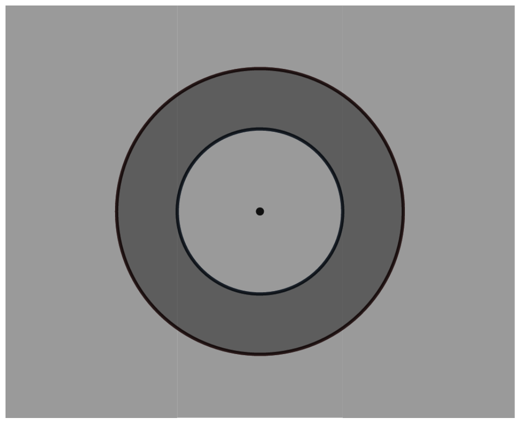


x varies from $-r$ to r:

$$
\begin{aligned}
\text{Volume } &= \int_{-r}^{r} \pi \big[R(x)\big]^2 dx \\
&= \int_{-r}^{r} \pi \big[\sqrt{r^2 - x^2}\big]^2 dx \\
&= \int_{-r}^{r} \pi (r^2 - x^2) dx \\
&= \int_{-r}^{r} (\pi r^2 - \pi x^2) dx \\
&= \left[\pi r^2 x - \frac{\pi x^3}{3}\right]_{-r}^{r} \\
&= \left(\pi r^2 r - \frac{\pi r^3}{3}\right) - \left(\pi r^2(-r) - \frac{\pi(-r)^3}{3}\right) \\
&= \left(\pi r^3 - \frac{\pi r^3}{3}\right) + \left(\pi r^3 - \frac{\pi r^3}{3}\right) \\
&= \frac{2\pi r^3}{3} + \frac{2\pi r^3}{3} \\
&= \frac{4\pi r^3}{3}.
\end{aligned}
$$

We have derived[2] the famous formula that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

- **The Washer Method:** This method is similar to the disk method, but it is for regions with gaps, which cause the cross-sections to not be disks but rather disks with disks removed (aka washers).



Here we have two concentric circles, and we need to know the area of the region within the larger one, but excluding the area inside the smaller one. If the larger circle has radius R and the smaller circle has radius r, then the area of the "washer" (the dark grey region) is $\pi R^2 - \pi r^2$. In solids of revolution for which each cross-section is a washer, the volume is given by

$$\text{Volume } = \int_a^b \left(\pi\big[R(x)\big]^2 - \pi\big[r(x)\big]^2\right) dx.$$

Consider the region bounded by the curves $y = x^2 + 1$ and $y = 2x + 1$. Let's find the volume of the solid obtained by rotating this region about the x-axis. Below is a picture of the region.

First we should figure out where out curves intersect. To do this, solve

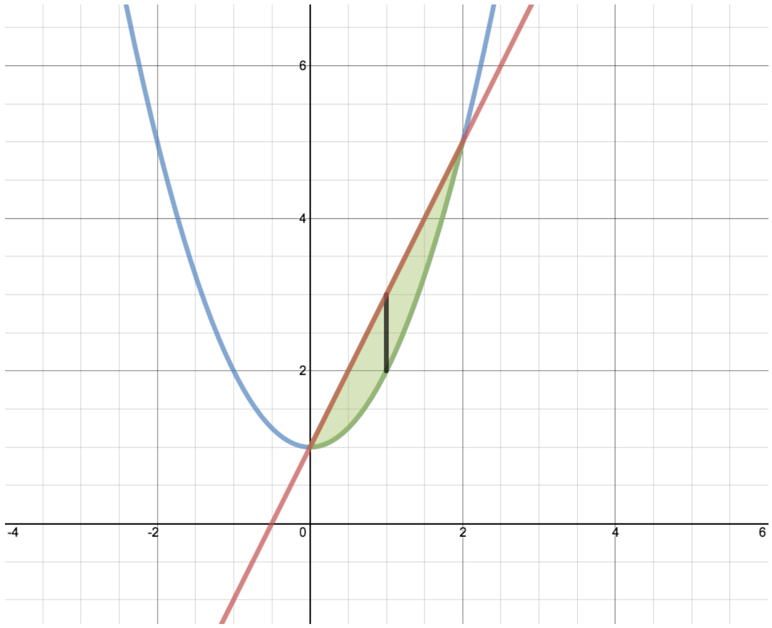
$$\begin{aligned} x^2 + 1 &= 2x + 1 \\ x^2 - 2x &= 0 \\ x(x-2) &= 0 \\ x &= 0, 2. \end{aligned}$$

As verified by the computer-drawn picture, the points of intersection happen at $x = 0, 2$. Consider the segment drawn in the region for a generic x.

[2]Of course in this whole enterprise we are heavily using the fact that the area of a circle of radius r is πr^2. You might be unsatisfied with *assuming* this, and you might want to derive it on your own. This can be computed as

$$\text{Area of circle } = 2\int_{-r}^{r} \sqrt{r^2 - x^2}dx.$$

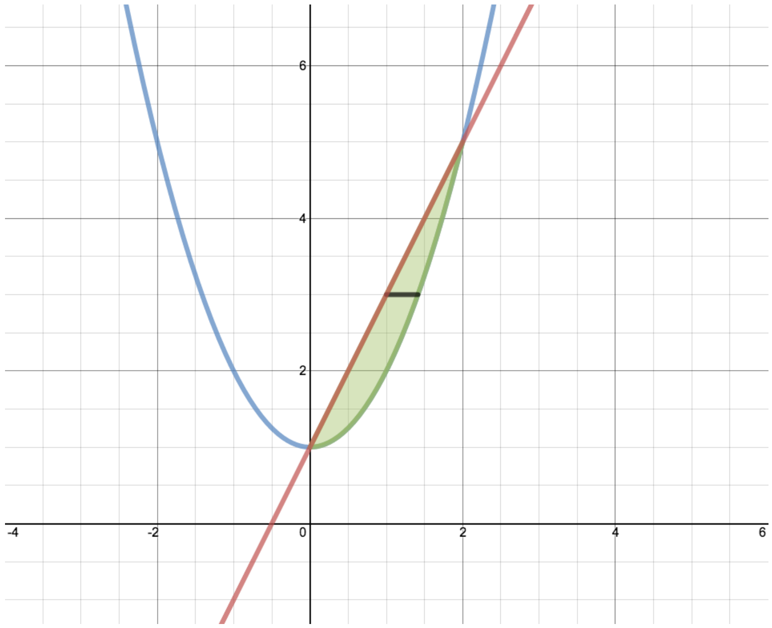
Unfortunately, this is a difficult integral which requires the technique of trigonometric substitution to solve. This method is typically covered in Calculu II.



When it rotates about the x-axis, it traces out a washer with outer radius $R(x) = 2x + 1$ and inner radius $r(x) = x^2 + 1$. So the volume is given by

$$\begin{aligned}
\text{Volume} &= \int_0^2 \left(\pi\left[R(x)\right]^2 - \pi\left[r(x)\right]^2\right) dx \\
&= \int_0^2 \left(\pi\left[2x+1\right]^2 - \pi\left[x^2+1\right]^2\right) dx \\
&= \int_0^2 \left(\pi\left[4x^2+4x+1\right] - \pi\left[x^4+2x^2+1\right]\right) dx \\
&= \int_0^2 \pi\left[4x + 2x^2 - x^4\right] dx \\
&= \pi\left[2x^2 + \frac{2x^3}{3} - \frac{x^5}{5}\right]_0^2 \\
&= \pi\left(2(2)^2 + \frac{2(2)^3}{3} - \frac{2^5}{5}\right) - \pi(0+0-0) \\
&= \pi\left(8 + \frac{16}{3} - \frac{32}{5}\right) \\
&= \pi\left(\frac{120}{15} + \frac{80}{15} - \frac{96}{15}\right) \\
&= \frac{104\pi}{15}.
\end{aligned}$$

Now let's rotate the same region about the y-axis to get a different solid, and compute its volume. We can still do the washer method, but the washers are oriented differently. Consider a generic line segment in the region that is perpendicular to the axis of rotation. As it rotates, it will



trace out a washer whose outer radius is given by the x-coordinate of the parabola and whose inner radius is given by the x coordinate of the line. We have one such washer for each value of y, so in this problem we will integrate with respect to y. We should express everything as a function of y. If we solve for x, the parabola $y = x^2 + 1$ becomes $x = \pm\sqrt{y-1}$. The positive square root $x = \sqrt{y-1}$ is the right half of the parabola, and this is what we what. Likewise we can take the line $y = 2x + 1$ and solve for x to get $x = \frac{y}{2} - \frac{1}{2}$. As said above, the outer radius is the x-coordinate of the parabola and the inner radius is the x-coordinate of the line, so

$$R(y) = \sqrt{y-1} \quad \text{and} \quad r(y) = \frac{y}{2} - \frac{1}{2}.$$

The volume is given by

$$\begin{aligned}
\text{Volume} &= \int_1^5 \left(\pi\big[R(y)\big]^2 - \pi\big[r(y)\big]^2\right) dy \\
&= \int_1^5 \left(\pi\big[\sqrt{y-1}\big]^2 - \pi\big[\frac{y}{2} - \frac{1}{2}\big]^2\right) dy \\
&= \int_1^5 \left(\pi\big[y-1\big] - \pi\big[\frac{y^2}{4} - \frac{y}{2} + \frac{1}{4}\big]\right) dy \\
&= \int_1^5 \pi\left(-\frac{y^2}{4} + \frac{3y}{2} - \frac{5}{4}\right) dy \\
&= \pi\left[-\frac{y^3}{12} + \frac{3y^2}{4} - \frac{5y}{4}\right]_1^5 \\
&= \pi\left(-\frac{5^3}{12} + \frac{3(5)^2}{4} - \frac{5(5)}{4}\right) - \pi\left(-\frac{1^3}{12} + \frac{3(1)^2}{4} - \frac{5(1)}{4}\right) \\
&= \pi\left(-\frac{125}{12} + \frac{75}{4} - \frac{25}{4}\right) - \pi\left(-\frac{1}{12} + \frac{3}{4} - \frac{5}{4}\right) \\
&= \pi\left(-\frac{125}{12} + \frac{50}{4}\right) - \pi\left(-\frac{1}{12} - \frac{2}{4}\right) \\
&= \pi\left(-\frac{125}{12} + \frac{150}{12}\right) - \pi\left(-\frac{1}{12} - \frac{6}{12}\right) \\
&= \pi\left(\frac{25}{12}\right) - \pi\left(-\frac{7}{12}\right) \\
&= \frac{32\pi}{12} = \frac{8\pi}{3}.
\end{aligned}$$

- **The Shell Method:** This is another method to compute volumes of revolution. It seems more complicated at first, but sometimes provides simpler solutions. Consider the region bounded by $y = x^2, y = -x^4$, and $x = 1$. This region will be revolved around the y-axis to form a solid of revolution. Let's find its volume. Rather than considering a segment of the region that is perpendicular to the axis of rotation, let's consider a segment parallel to the axis of rotation.

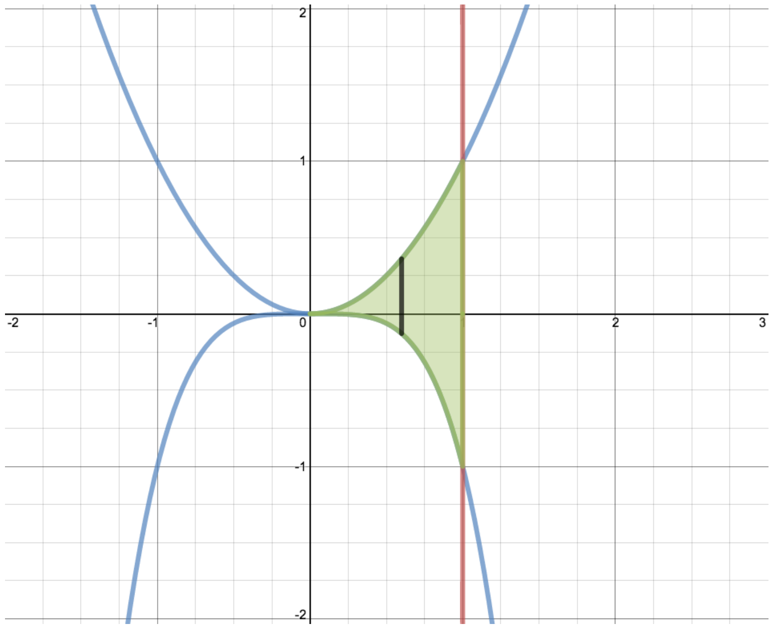
  When we rotate this segment around the y-axis, it traces out a cylindrical shell. The total volume of our solid can be obtained by integrating the surface area of all the cylindrical shells as x varies from 0 to 1. The surface area of a cylindrical shell is

  $$\text{Area of Shell } = 2\pi(\text{Shell Radius})(\text{Shell Height}).$$

  Thus, the volume can be calculated as

  $$\text{Volume } = \int_0^1 2\pi(\text{Shell Radius})(\text{Shell Height})dx.$$

  For the shell at the point x, we need to express the shell radius and shell height in terms of x. It helps to visualize the segment rotating around the y-axis to determine the radius and height of the shell. The radius is just the distance from the segment to the y-axis, which is x. The shell height



is the length of the segment, which is the difference in y-coordinates of the two functions: $x^2 - (-x^4) = x^2 + x^4$. So we calculate

$$\begin{aligned}
\text{Volume} &= \int_0^1 2\pi(\text{Shell Radius})(\text{Shell Height})dx \\
&= \int_0^1 2\pi(x)(x^2 + x^4)dx \\
&= 2\pi \int_0^1 (x^3 + x^5)dx \\
&= 2\pi \left[\frac{x^4}{4} + \frac{x^6}{6}\right]_0^1 \\
&= 2\pi \left(\frac{1^4}{4} + \frac{1^6}{6}\right) - 2\pi(0 + 0) \\
&= 2\pi \left(\frac{5}{12}\right) \\
&= \frac{5\pi}{6}.
\end{aligned}$$

This problem could also be solved using the washer method. However more work is required because the formula defining the curve which is the left boundary changes depending on whether we are above or below the x-axis. So we would have to divide the region into two separate regions, and compute the volume using two integrals. Try it and make sure you get $\frac{5\pi}{6}$ as the answer.

The shell method can also be done for regions which are rotated around the x-axis. Here you should draw the segment which will generate a shell as horizontal so that it is parallel to the axis of rotation. This will help you visualize the shell. Then the integration will have to be done with respect to y.